§ 5. Continuity of Functions:

Definition: A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if:

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Translation:

$$\lim_{x \to a} f(x) = f(a)$$

"where the outputs look to be going towards as inputs go to a" = "actual output at $x = a$"

Note: From both sides!

Remark: For f to be continuous at a, we must have:

1) $f(a)$ is defined.

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$ ← Need both sides!

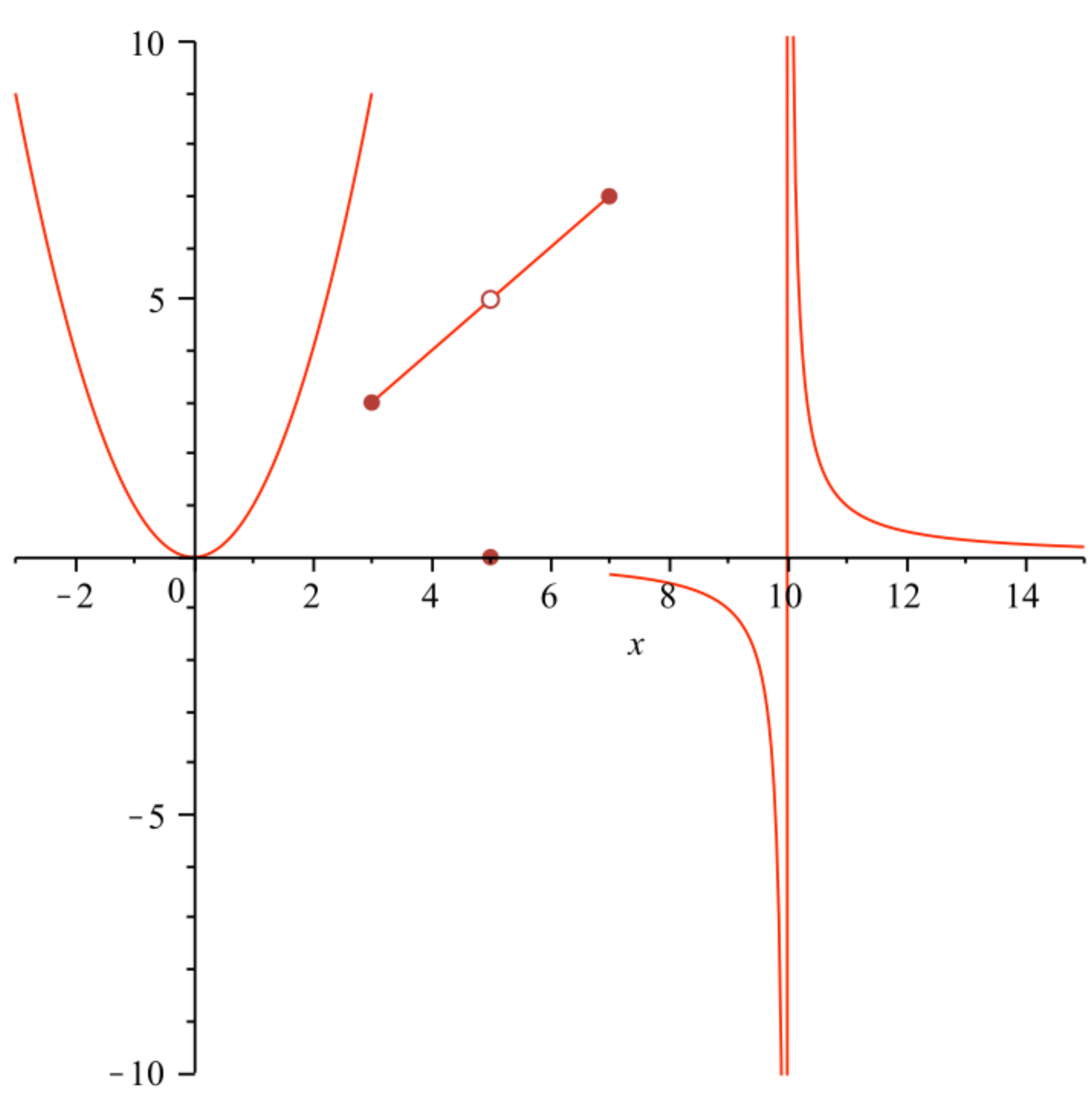
Definition: If a function f is defined near a, we say f is discontinuous at a if f is not continuous at a.

Remark: There are many ways this can happen. Refer back to remark on page 1.

Example 2 Consider the graph shown below of the function

$$k(x) = \begin{cases} x^2 & -\infty < x < 3 \\ x & 3 \le x < 5 \\ 0 & x = 5 \\ x & 5 < x \le 7 \\ \frac{1}{x-10} & x > 7 \end{cases}$$

Where is the function discontinuous and why?



Use this space to answer the example.

Definitions:

1) We say f has a removable discontinuity at a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

2) If f has a vertical asymptote at a, we say f has an infinite discontinuity.

3) We say f has a jump discontinuity if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, but are not equal.

(i.e., the graph "jumps").

Exercise: Classify the discontinuities in the previous example.

Definitions:

1) A function f is continuous from the right at a number a if

$$\boxed{\lim_{x \to a^+} f(x) = f(a)}$$

2) A function f is continuous from the left at a number a if

$$\boxed{\lim_{x \to a^-} f(x) = f(a)}$$

Example 3 Consider the function $k(x)$ in example 2 above. At which of the following x-values is $k(x)$ continuous from the right?

$$x = 0, \quad x = 3, \quad x = 5, \quad x = 7, \quad x = 10.$$

At which of the above x-values is $k(x)$ continuous from the left?

Definition: A function f is continuous on an interval if it is continuous at every number in the interval. (at the endpoints, you only need left or right continuous if you function is only defined on one side of the endpoint).

Example Consider the function $k(x)$ in example 2 above.
(a) On which of the following intervals is $k(x)$ continuous?

$$(-\infty, 0], \quad (-\infty, 3), \qquad [3, 7].$$

(b) Fill in the missing endpoints and brackets which give the largest intervals on which $k(x)$ is continuous.

$(-\infty,$ $\qquad\qquad$ $(5,$

Example Let

$$m(x) = \begin{cases} cx^2 + 1 & x \geq 2 \\ 10 - x & x < 2 \end{cases}$$

For which value of c is $m(x)$ a continuous function?

Catalogue of Continuous Functions:

1) Polynomials are continuous on all of $\mathbb{R}$ (at all numbers).

2) Rational functions $\left(f(x) = \frac{P(x)}{Q(x)} \right)$ are continuous everywhere they are defined (i.e. at all points where $Q(x) \neq 0$).

3) Root functions $\left(f(x) = \sqrt[n]{x} \right)$ are continuous everywhere they are defined
(i.e. if n is even, they are continuous on $[0, \infty)$
if n is odd, they are continuous on all of $\mathbb{R}$ (at all numbers)).

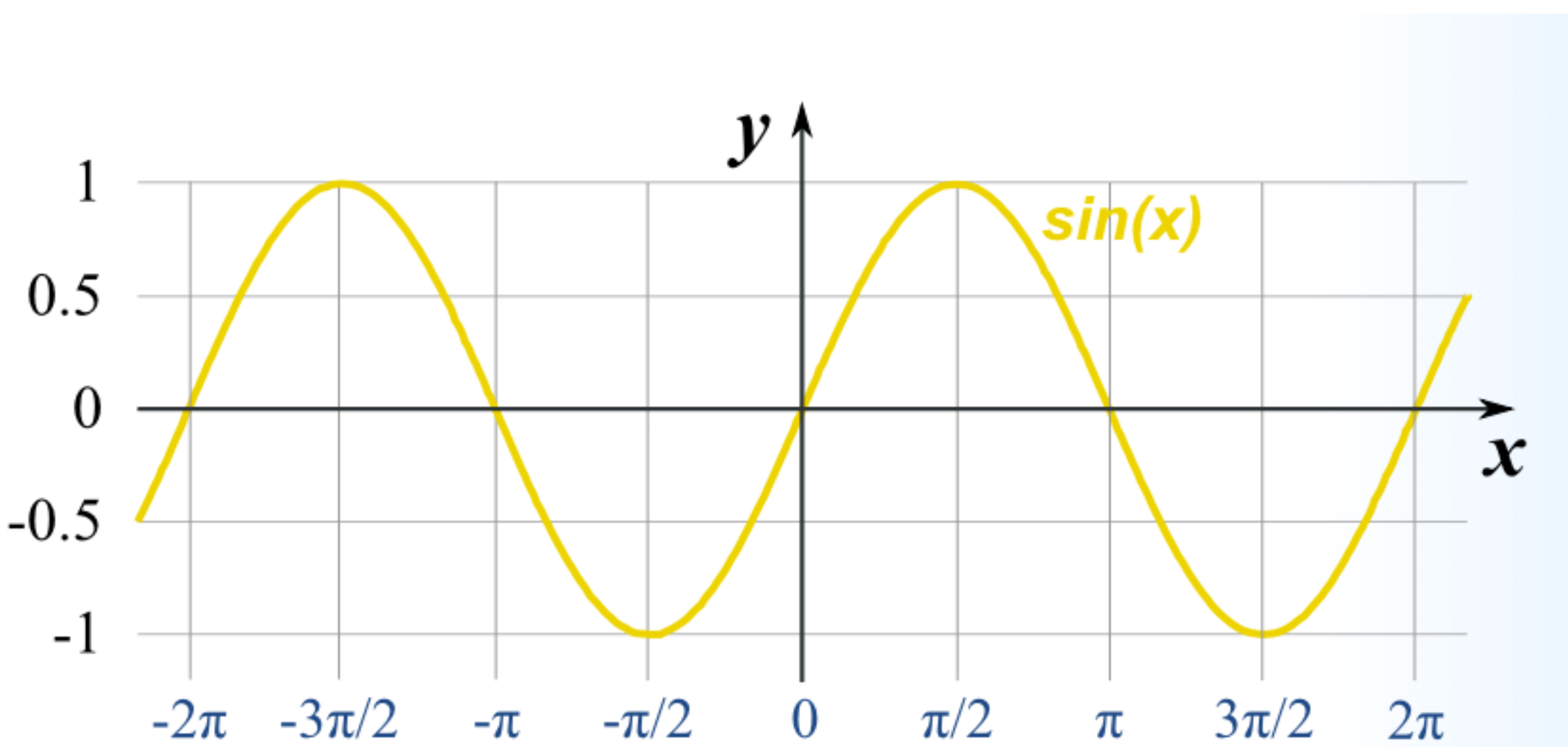
4) Trigonometric Functions are continuous on all of their domains.

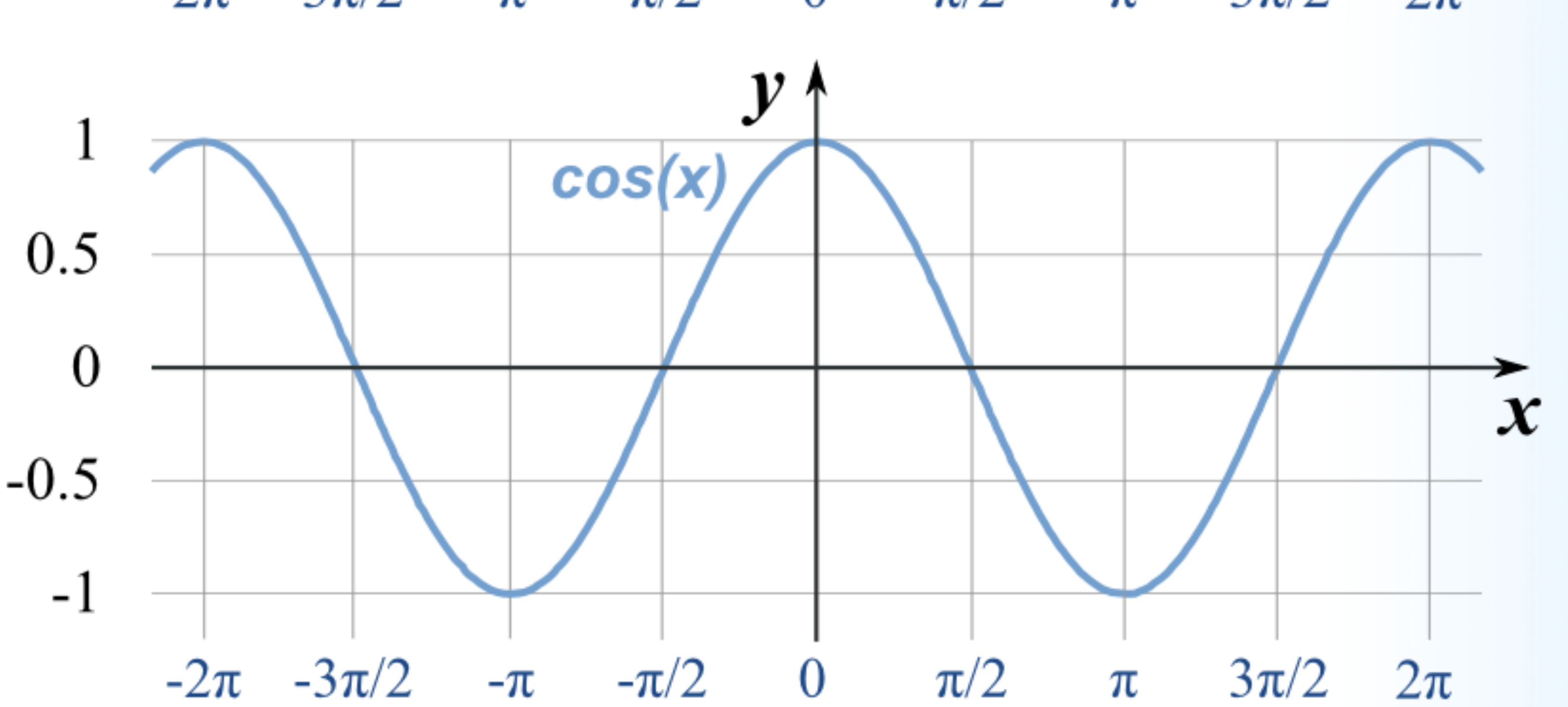
e.g. sin and cos are continuous on all of $\mathbb{R}$.

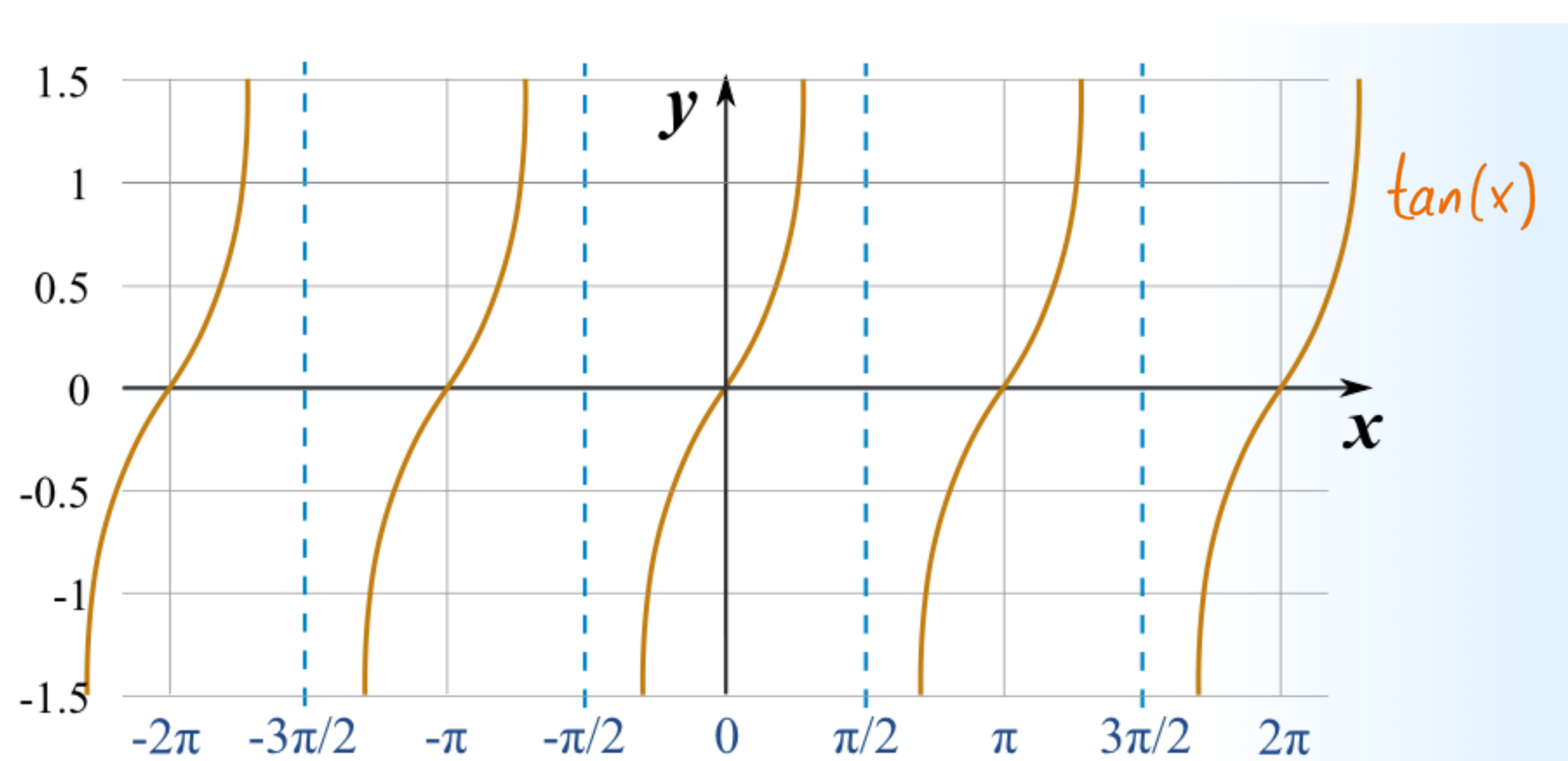
tan is continuous everywhere it is defined (i.e. all points where $\cos(x) \neq 0$)

Remark: $\cos(x) = 0$ for $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

sec, csc and cot are all continuous on their domains. We can see this from their graphs.







Combinations of Continuous Functions:

Theorem: If f and g are continuous at a and c is a constant, then:

1) $f+g$ is continuous at a
2) $f-g$ is continuous at a
3) cf is continuous at a
4) fg is continuous at a
5) $\frac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Remark: This theorem follows from the theorem on page 1 of the previous lecture (Laws of Limits).

Remark: Using the above results, we now know that combinations of Polynomial, Rational, Root and Trigonometric functions using $+, -, \cdot, \div$ are continuous on their domains.

multiply

Examples: Write down the domains of the functions described by the following algebraic representations and justify why they are continuous on their domains:

1) $$g(x) = \frac{(x^2+3)^3}{x-10}$$

Solution: $g : \mathbb{R} \setminus \{10\} = (-\infty, 10) \cup (10, \infty) \longrightarrow \mathbb{R}$

It is continuous on it's domain as it is a rational function.

2) $$K(x) = \sqrt[3]{x}\left(\frac{x^2-3}{x^2+1}\right) - \frac{1}{x-27}$$

Example: Removable Discontinuity Recall that last day we found $\lim_{x\to 0} x^2 \sin(1/x)$ using the squeeze theorem. What is the limit?

Does the function

$$n(x) = \begin{cases} x^2 \sin(1/x) & x > 0 \\ x^2 \sin(1/x) & x < 0 \end{cases}$$

have a removable discontinuity at zero?
(in other words can I define the function to have a value at $x = 0$ making a continuous function?)

$$n_1(x) = \begin{cases} x^2 \sin(1/x) & x > 0 \\ ? & x = 0 \\ x^2 \sin(1/x) & x < 0 \end{cases}$$

Composing Continuous Functions:

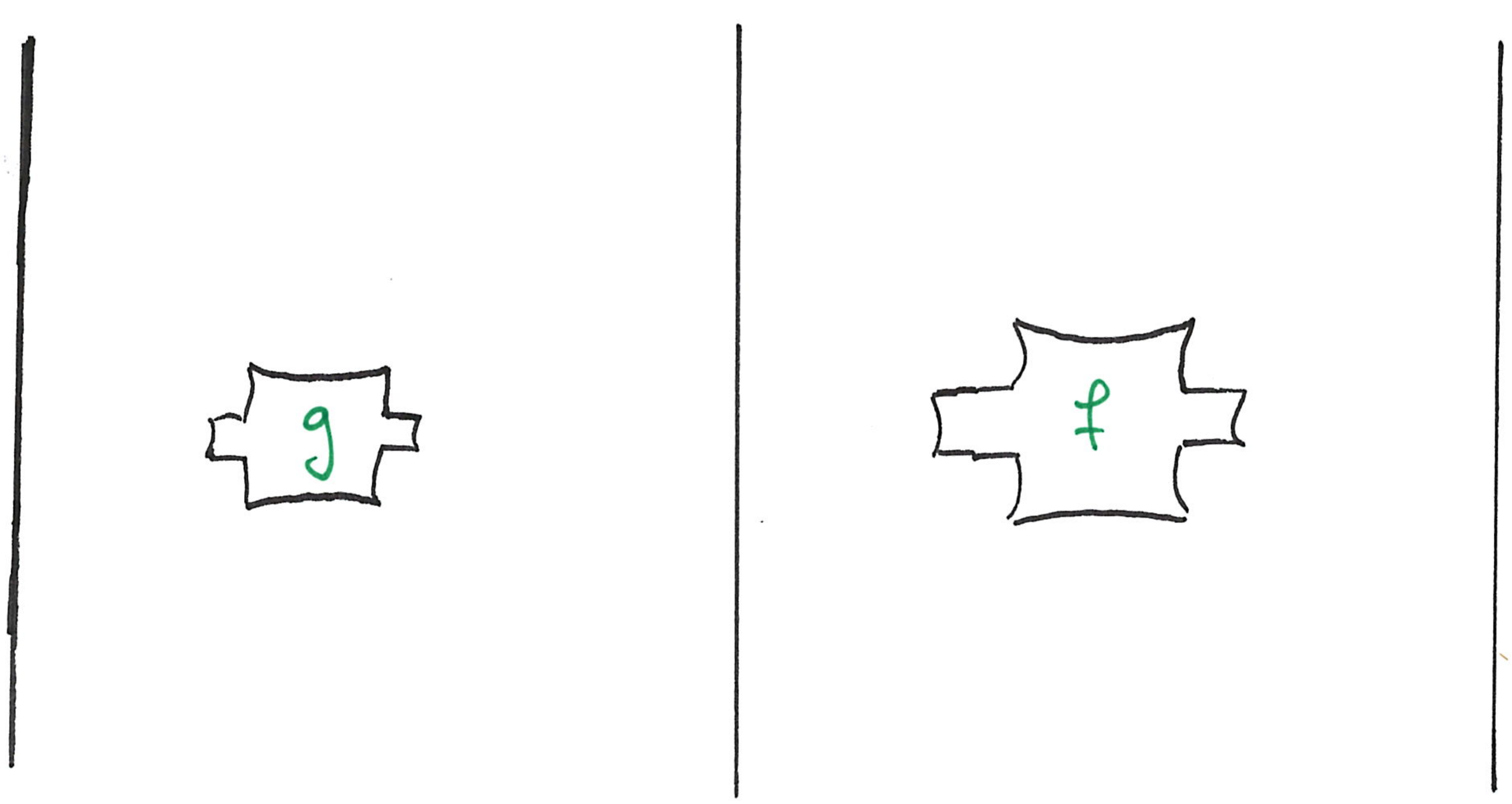
Theorem: If g is continuous at a, and f is continuous at $g(a)$, then their composition $(f \circ g)(x) := f(g(x))$ is continuous at a:

$$\boxed{\lim_{x \to a} (f \circ g)(x) = (f \circ g)(a)}$$

i.e.

$$\boxed{\lim_{x \to a} f(g(x)) = f(g(a))}$$

Picture:



Example: Evaluate the following limit:

$$\lim_{x \to 0} \sin\left(\frac{x^2+\pi}{x^4+1}\right)$$

(a) What is the domain of the function described by

$$h(x) = \sin\left(\frac{x^2+\pi}{x^4+1}\right)$$

(b) Where is h continuous?

Theorem: (Intermediate Value Theorem)

Suppose that f is continuous on $[a, b]$ and let r be any number between $f(a)$ and $f(b)$. Then there exists $c \in (a, b)$ $(a < c < b)$ such that $f(c) = r$.

Translation: If at $x = a$, we are at height H_1 (i.e. $f(a) = H_1$) and at $x = b$, we are at height H_2 (i.e. $f(b) = H_2$) we must move through all heights between H_1 and H_2 as our inputs move from a to b.

Picture:

Remark: This corresponds to the graph of the function "being drawn without lifting the pen from the paper" or "having no holes or gaps".

Examples:

1) Show $x^2 - 3$ has a root between 0 and 2.

2) Show $\cos(x) = x^2$ has a solution.

Hint: Consider $h(x) = \cos(x) - x^2$

Extra Examples, Please attempt the following problems before looking at the solutions

Example Which of the following functions are continuous on the interval $(0, \infty)$:

$$f(x) = \frac{x^3 + x - 1}{x + 2}, \qquad g(x) = \frac{x^2 + 3}{\cos x}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}, \qquad k(x) = |\sin x|.$$

Example Which of the following functions have a removable discontinuity at $x = 2$?:

$$f(x) = \frac{x^3 + x - 1}{x - 2}, \qquad g(x) = \frac{x^2 - 4}{x - 2}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}.$$

Example Find the domain of the following function and use Theorems 1, 2 and 3 to show that it is continuous on its domain:

$$k(x) = \frac{\sqrt[3]{\cos x}}{x - 10}.$$

Example Evaluate the following limits:

$$\lim_{x \to \pi} \sqrt[3]{2 + \cos x} \qquad \lim_{x \to \frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}}$$

Example What is the domain of the following function and what are the (largest) intervals on which it is continuous?

$$g(x) = \frac{1}{\sqrt{1 - \sqrt{x}}}.$$

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\sqrt[3]{x} = 1 - x \qquad (0, 1).$$

Solutions

Example Which of the following functions are continuous on the interval $(0, \infty)$:

$$f(x) = \frac{x^3 + x - 1}{x + 2}, \qquad g(x) = \frac{x^2 + 3}{\cos x}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}, \qquad k(x) = |\sin x|.$$

Since $f(x)$ is a rational function, it is continuous everywhere except at $x = -2$, Therefore it is continuous on the interval $(0, \infty)$.
By Theorem 2 and the continuity of polynomials and trigonometric functions, $g(x)$ is continuous except where $\cos x = 0$. Since $\cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, we have $g(x)$ is not continuous on $(0, \infty)$.
By theorems 2 and 3, $h(x)$ is continuous everywhere except at $x = 2$. In fact $x = 2$ is not in the domain of this function. Hence the function is not continuous on the interval $(0, \infty)$.
Since $k(x) = |\sin x| = F(G(x))$, where $G(x) = \sin x$ and $F(x) = |x|$, we have that $k(x)$ is continuous everywhere on its domain since both F and G are both continuous everywhere on their domains. Its not difficult to see that the domain of k is all real numbers, hence k is continuous everywhere. (What does its graph look like?)

Example Which of the following functions have a removable discontinuity at $x = 2$?:

$$f(x) = \frac{x^3 + x - 1}{x - 2}, \qquad g(x) = \frac{x^2 - 4}{x - 2}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}.$$

$\lim_{x\to 2} f(x)$ does not exist, since $\lim x \to 2(x^3 + x - 1) = 9$ and $\lim x \to 2(x - 2) = 0$. Therefore the discontinuity is not removable.
$\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x + 2) = 4$. Therefore the discontinuity at $x = 2$ is removable by defining a piecewise function:

$$g_1(x) = \begin{cases} g(x) & x \neq 2 \\ 4 & x = 2 \end{cases}$$

$\lim_{x\to 2} h(x)$ does not exist, since $\lim_{x\to 2}(\sqrt{x^2 + 1}) = \sqrt{(5)}$ and $\lim x \to 2(x - 2) = 0$. Therefore the discontinuity is not removable.

Example Find the domain of the following function and use Theorems 1, 2 and 3 to show that it is continuous on its domain:

$$k(x) = \frac{\sqrt[3]{\cos x}}{x - 10}.$$

The domain of this function is all values of x except $x = 10$, since $\cos x$ is defined everywhere as is the cubed root function. Theorem 1 says that the cosine function is continuous everywhere and theorem 3 says that $f(x) = \sqrt[3]{\cos x}$ is continuous for all real numbers since the cubed root function is continuous everywhere. Now we see from Theorem 2 that $k(x) = \frac{f(x)}{g(x)}$ is continuous everywhere except where $g(x) = x - 10 = 0$, that is at $x = 10$.

Example Evaluate the following limits:

$$\lim_{x\to\pi} \sqrt[3]{2 + \cos x} \qquad\qquad \lim_{x\to\frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}}$$

Since $G(x) = 2 + \cos x$ and $F(x) = \sqrt[3]{x}$ are continuous everywhere, we have $F(Gx))$ is continuous on its domain and we can calculate the first limit by evaluation:

$$\lim_{x\to\pi} \sqrt[3]{2 + \cos x} = \sqrt[3]{2 + \cos \pi} = \sqrt[3]{2 - 1} = 1.$$

As above, we have $\sqrt[3]{\sin x}$ is continuous on its domain, therefore $\lim_{x\to\frac{\pi}{2}} \sqrt[3]{\sin x} = \sqrt[3]{\sin\frac{\pi}{2}} = 1$. Since $\lim_{x\to\frac{\pi}{2}}(x - \frac{\pi}{2}) = 0$, we have $\frac{\sqrt[3]{\sin x}}{x-\frac{\pi}{2}}$ approaches ∞ in absolute value as x approaches $\frac{\pi}{2}$. As $x \to \frac{\pi}{2}^-$, $\sin(x) > 0$, hence $\sqrt[3]{\sin x} > 0$. As $x \to \frac{\pi}{2}^-$, $x - \frac{\pi}{2} < 0$, therefore the quotient has negative values and

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}} = -\infty.$$

Example What is the domain of the following function and what are the (largest) intervals on which it is continuous?

$$g(x) = \frac{1}{\sqrt{1-\sqrt{x}}}.$$

The domain of this function is all x where $\sqrt{1-\sqrt{x}} \neq 0$, i.e. all x where $x \neq 1$. By theorems 3 and 2, the function is continuous everywhere on its domain, therefore it is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$.

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\sqrt[3]{x} = 1 - x \qquad (0, 1).$$

Let $g(x) = \sqrt[3]{x} - 1 + x$. We have $g(0) = -1 < 0$ and $g(1) = 1 > 0$. therefore by the intermediate value theorem, there is some number c with $0 < c < 1$ for which $g(c) = 0$. That is

$$\sqrt[3]{c} = 1 - c$$

as desired.