§ 4. Laws of Limits :

The following theorem, in some sense, shows us that limits behave how we expect them to:

Theorem : Say $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist (i.e. they are finite numbers), and $c \in \mathbb{R}$ is a constant. Then:

1) $$\boxed{\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)}$$

("Limit of sum = Sum of limits")

2) $$\boxed{\lim_{x \to a} \left(f(x) - g(x) \right) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)}$$

("Limit of difference = Difference of limits")

3) $$\boxed{\lim_{x \to a} \left(f(x) g(x) \right) = \left(\lim_{x \to a} f(x) \right) \left(\lim_{x \to a} g(x) \right)}$$

("Limit of product = Product of limits")

4) $$\boxed{\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)}$$

("You can pull constants past limits.")

5) If $\lim_{x \to a} g(x) \neq 0$, then :

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}}$$

("Limit of a quotient = Quotient of limits.")

Example If I am given that

$$\lim_{x \to 2} f(x) = 2, \qquad \lim_{x \to 2} g(x) = 5, \qquad \lim_{x \to 2} h(x) = 0.$$

find the limits that exist (are a finite number):

$$(a) \quad \lim_{x \to 2} \frac{2f(x) + h(x)}{g(x)}$$

$$(b) \quad \lim_{x \to 2} \frac{f(x)}{h(x)}$$

$$(c) \quad \lim_{x \to 2} \frac{f(x)h(x)}{g(x)}$$

(a) As $\lim_{x \to 2} g(x) = 5 \neq 0$, then:

$$\lim_{x \to 2} \frac{2f(x) + h(x)}{g(x)} = \lim_{x \to 2} \left(\frac{2f(x)}{g(x)} + \frac{h(x)}{g(x)} \right)$$

$$\overset{(1)}{=} \lim_{x \to 2} \frac{2f(x)}{g(x)} + \lim_{x \to 2} \frac{h(x)}{g(x)}$$

$$= \frac{2 \lim_{x \to 2} f(x)}{\lim_{x \to 2} g(x)} + \frac{\lim_{x \to 2} h(x)}{\lim_{x \to 2} g(x)}$$

$$= \frac{2(2)}{5} + \frac{0}{2}$$

$$= \frac{4}{5}$$

(b)

(c)

Remark: If $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x) = b$, where b is finite and $b \neq 0$, then we can make the size (absolute value) of $\frac{f(x)}{g(x)}$ arbitrarily large by taking x close to a. Hence, the $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist (DNE).

1) If $\frac{f(x)}{g(x)}$ is always positive for x sufficiently close to a, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \infty$$

2) If $\frac{f(x)}{g(x)}$ is always negative for x sufficiently close to a, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty$$

3) If neither of the above is the case, we just say $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist (DNE).

Examples: 1) $\lim_{x \to 0} x = 0$, $\lim_{x \to 0} 1 = 1$,

$\lim_{x \to 0} \frac{1}{x}$ DNE.

2) $\lim\limits_{x \to 0} \frac{1}{x^2} =$ ______

More powerful laws of limits can be derived using the above laws 1-5 and our knowledge of some basic functions. The following can be proven reasonably easily (we are still assuming that c is a constant and $\lim_{x\to a} f(x)$ exists);

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer (we see this using rule 4 repeatedly).
7. $\lim_{x\to a} c = c$, where c is a constant (easy to prove from definition of limit and easy to see from the graph, $y = c$).
8. $\lim_{x\to a} x = a$, (follows easily from the definition of limit)
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer (this follows from rules 6 and 8).
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer and $a > 0$ if n is even. (proof needs a little extra work and the binomial theorem)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ assuming that the $\lim_{x\to a} f(x) > 0$ if n is even. (We will look at this in more detail when we get to continuity)

Example Evaluate the following limits and justify each step:

(a) $\lim_{x \to 3} \frac{x^3+2x^2-x+1}{x-1}$

(b) $\lim_{x \to 1} \sqrt[3]{x+1}$

Limits of Polynomials and Rational Functions:

- Direct Substitution Property: If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a) \quad .$$

NB

- If f is a rational function with $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with $Q(a) = 0$, then we must use algebra to attempt to find the limit (i.e. factorise, etc.).

Example Determine if the following limits are finite, equal to $\pm\infty$ or D.N.E. and are not equal to $\pm\infty$:

(a) $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

(b) $\lim_{x\to 1^-} \frac{x^2-x-6}{x-1}$.

(c) Which of the following is true:

1. $\lim_{x\to 1} \frac{x^2-x-6}{x-1} = +\infty$, 2. $\lim_{x\to 1} \frac{x^2-x-6}{x-1} = -\infty$,

3. $\lim_{x\to 1} \frac{x^2-x-6}{x-1}$ D.N.E. and is not $\pm\infty$,

Example Evaluate the limit (finish the calculation)

$$\lim_{h \to 0} \frac{(3+h)^2 - (3)^2}{h}.$$

Example Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sqrt{x^2+25} - 5}{x^2}.$$

Recall also our observation from the last day which can be proven rigorously from the definition (this is good to keep in mind when dealing with piecewise defined functions):

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Example Evaluate the limit if it exists:

$$\lim_{x \to -2} \frac{3x+6}{|x+2|}$$

The following theorems help us calculate some important limits by comparing the behavior of a function with that of other functions for which we can calculate limits:

Theorem: If $f(x) \leq g(x)$ for x sufficiently close to a (not necessarily at a), and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then:

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Picture:

Theorem: (Squeeze Theorem)

If $f(x) \le g(x) \le h(x)$ for x sufficiently close to a (but not necessarily at a) and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

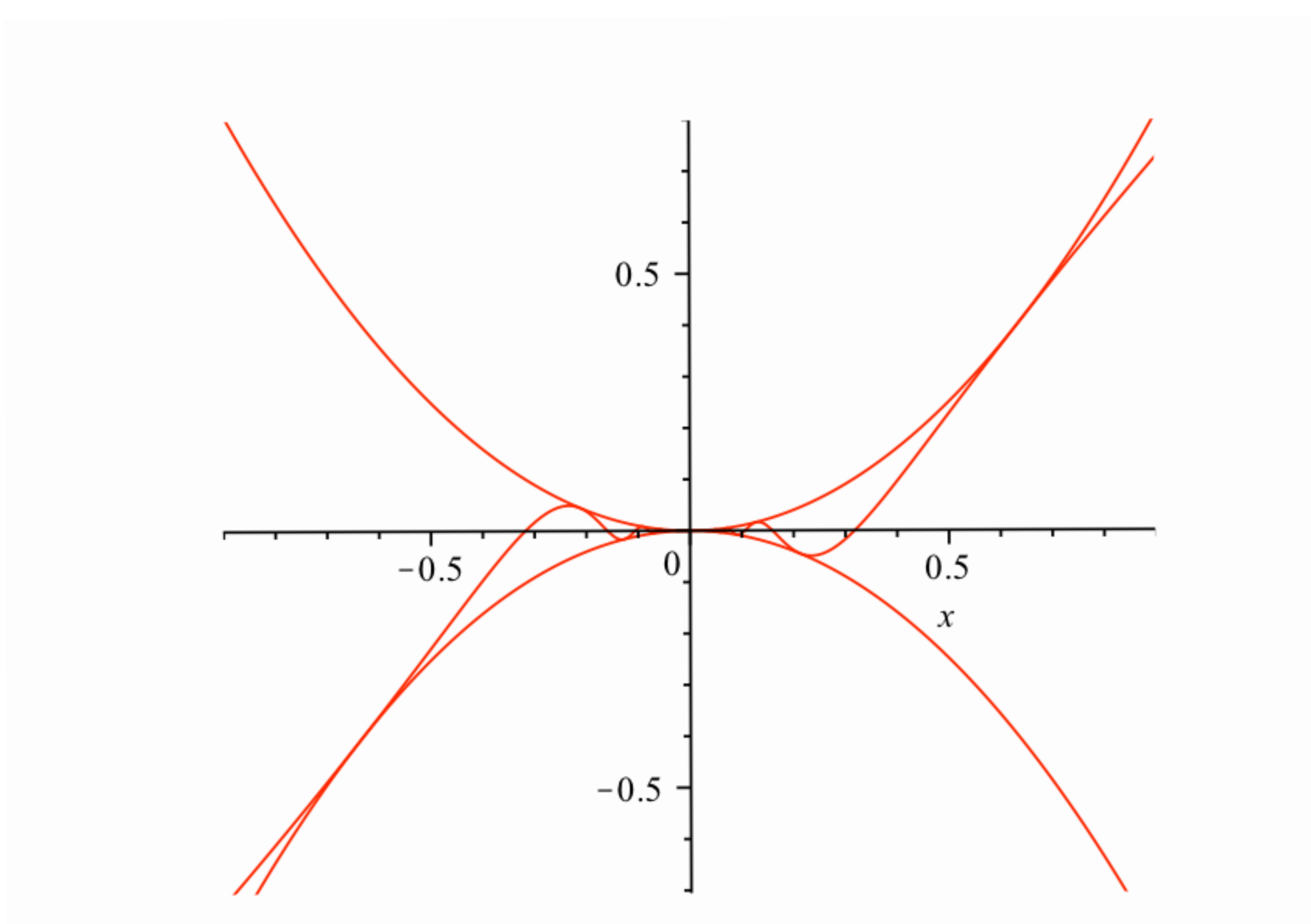
then

$$\lim_{x \to a} g(x) = L$$

Picture:

Example: Calculate $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Hint: $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$.



Example Decide if the following limit exists and if so find its values:

$\lim_{x \to o} x^{100} \cos^2(\pi/x)$

Extra Examples, attempt the problems before looking at the solutions

Decide if the following limits exist and if a limit exists, find its value.

(1) $\lim_{x\to 1} x^4 + 2x^3 + x^2 + 3$

(2) $\lim_{x\to 2} \frac{x^2-3x+2}{(x-2)^2}$.

(3) $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$.

(4) $\lim_{x\to 0} \frac{|x|}{x^2+x+10}$.

(5) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

(6) If $2x \le g(x) \le x^2 - x + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

(7) Determine if the following limit is finite, $\pm\infty$ or D.N.E. and is not $\pm\infty$.

$$\lim_{x\to 1^-} \frac{(x-3)(x+2)}{(x-1)(x-2)}.$$

Extra Examples, attempt the problems before looking at the solutions

Decide if the following limits exist and if a limit exists, find its value.

(1) $\lim_{x\to 1} x^4 + 2x^3 + x^2 + 3$

Since this is a polynomial function, we can calculate the limit by direct substitution:

$$\lim_{x\to 1} x^4 + 2x^3 + x^2 + 3 = 1^4 + 2(1)^3 + 1^2 + 3 = 7.$$

(2) $\lim_{x\to 2} \frac{x^2-3x+2}{(x-2)^2}$.

This is a rational function, where both numerator and denominator approach 0 as x approaches 2. We factor the numerator to get

$$\lim_{x\to 2} \frac{x^2-3x+2}{(x-2)^2} = \lim_{x\to 2} \frac{(x-1)(x-2)}{(x-2)^2}$$

After cancellation, we get

$$\lim_{x\to 2} \frac{(x-1)(x-2)}{(x-2)^2} = \lim_{x\to 2} \frac{(x-1)}{(x-2)}.$$

Now this is a rational function where the numerator approaches 1 as $x \to 2$ and the denominator approaches 0 as $x \to 2$. Therefore

$$\lim_{x\to 2} \frac{(x-1)}{(x-2)}$$

does not exist.

We can analyze this limit a little further, by checking out the left and right hand limits at 2. As x approaches 2 from the left, the values of $(x-1)$ are positive (approaching a constant 1) and the values of $(x-2)$ are negative (approaching 0). Therefore the values of $\frac{(x-1)}{(x-2)}$ are negative and become very large in absolute value. Therefore

$$\lim_{x\to 2^-} \frac{(x-1)}{(x-2)} = -\infty.$$

Similarly, you can show that

$$\lim_{x\to 2^-} \frac{(x-1)}{(x-2)} = +\infty,$$

and therefore the graph of $y = \frac{(x-1)}{(x-2)}$ has a vertical asymptote at $x = 2$.
(check it out on your calculator)

(3) $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$.

Let $f(x) = \frac{1}{x} - \frac{1}{|x|}$. We write this function as a piecewise defined function:

$$f(x) = \begin{cases} \frac{1}{x} - \frac{1}{x} = 0 & x > 0 \\ \frac{1}{x} + \frac{1}{x} = \frac{2}{x} & x \le 0 \end{cases}.$$

$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$ exists only if the left and right hand limits exist and are equal. $\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{|x|}) = \lim_{x\to 0^+} 0 = 0$ and $\lim_{x\to 0^-}(\frac{1}{x} - \frac{1}{|x|}) = \lim_{x\to 0^-} \frac{2}{x} = -\infty$. Since the limits do not match, we have

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|}) \;\; D.N.E.$$

(4) $\lim_{x\to 0} \frac{|x|}{x^2+x+10}$.

Since $\lim_{x\to 0} x^2 + x + 10 = 10 \neq 0$, we have

$$\lim_{x\to 0} \frac{|x|}{x^2+x+10} = \frac{\lim_{x\to 0}|x|}{\lim_{x\to 0}(x^2+x+10)} = \frac{\lim_{x\to 0}|x|}{10}.$$

Now

$$|x| = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases}.$$

Clearly $\lim_{x\to 0^+}|x| = 0 = \lim_{x\to 0^-}|x|$. Hence $\lim_{x\to 0}|x| = 0$ and

$$\lim_{x\to 0} \frac{|x|}{x^2+x+10} = \frac{\lim_{x\to 0}|x|}{10} = \frac{0}{10} = 0.$$

(5) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

Since $\lim_{h\to 0}\sqrt{4+h} - 2 = 0 = \lim_{h\to 0} h$, we cannot determine whether this limit exists or not from the limit laws without some transformation. We have

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{(\sqrt{4+h})^2 - 4)}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{(\sqrt{4+h}+2)} = \frac{1}{4}.$$

(6) If $2x \leq g(x) \leq x^2 - x + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

We use the Sandwich theorem here. Since $2x \leq g(x) \leq x^2 - x + 2$, we have

$$\lim_{x\to 1} 2x \leq \lim_{x\to 1} g(x) \leq \lim_{x\to 1}(x^2 - x + 2),$$

therefore

$$2 \leq \lim_{x\to 1} g(x) \leq 2$$

and hence

$$\lim_{x\to 1} g(x) = 2.$$

(7) Determine if the following limit is finite, $\pm\infty$ or D.N.E. and is not $\pm\infty$.

$$\lim_{x \to 1^-} \frac{(x-3)(x+2)}{(x-1)(x-2)}.$$

Let $P(x) = (x-3)(x+2)$ and $Q(x) = (x-1)(x-2)$. We have $P(1) = -6 \neq 0$ and $Q(1) = 0$. Therefore the values of $\frac{P(x)}{Q(x)} = \frac{(x-3)(x+2)}{(x-1)(x-2)}$ get larger in absolute value as x approaches 1.

As x approaches 1 from the left, $(x-3) < 0$, $(x-2) < 0$, $(x-1) < 0$, and $(x+2) > 0$, therefore the quotient $\frac{(x-3)(x+2)}{(x-1)(x-2)} < 0$ as x approaches 1 from the left and therefore

$$\lim_{x \to 1^-} \frac{(x-3)(x+2)}{(x-1)(x-2)} = -\infty.$$