§ 3. Limits of Functions:

Idea: Roughly speaking, when we are considering limits of functions, we are trying to determine the behaviour of the outputs as the inputs approach a particular value.

Example: $f : \mathbb{R} \longrightarrow \mathbb{R}$

$: x \longmapsto x^2$ (i.e. $f(x) = x^2$)

Consider the behaviour of the outputs as the inputs approach 3:

i.e. Consider the behaviour of $f(x)$ as x approachs 3:

x	$f(x) = x^2$
2	4
2.5	6.25
2.95	8.70
2.995	8.97
2.999	8.99

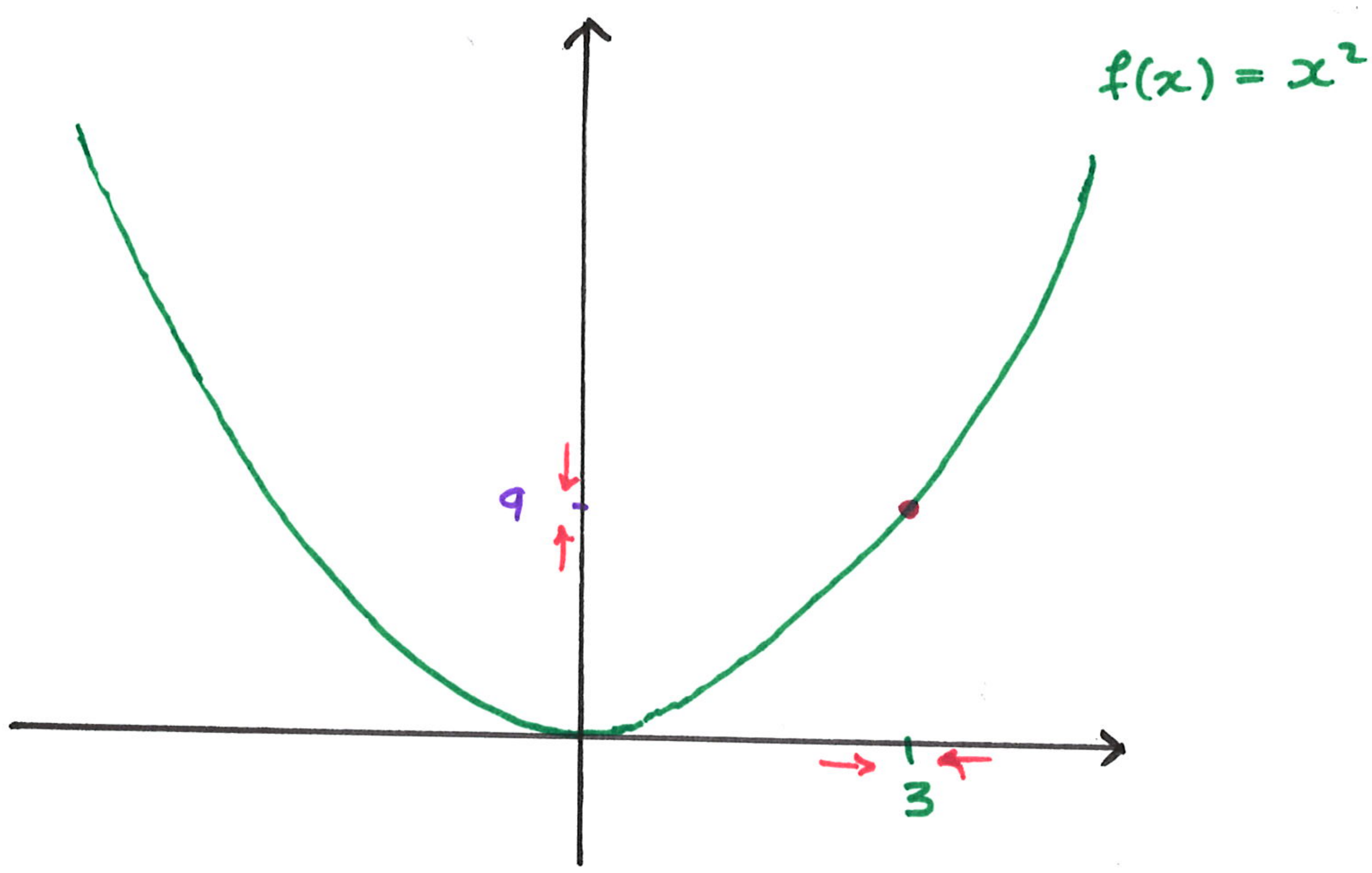
x	$f(x) = x^2$
4	16
3.5	12.25
3.01	9.06
3.005	9.03
3.001	9.006

We see as our inputs, x, approach 3, our outputs, $f(x)$, approach ______ .

We also say "$f(x)$ tends to 9 as x tends to 3". Or, with notation:

$$f(x) \longrightarrow 9 \quad \text{as} \quad x \longrightarrow 3$$

We can also see this from the graph:



As our inputs (widths) approach 3, the outputs (heights) approach 9.

Remark: Notice that we get the same value (9) if we approach from the left of 3, or the right of 3.

Definition: We say the limit as x approaches a of $f(x)$ equals L, and write

$$\boxed{\lim_{x \to a} f(x) = L}$$

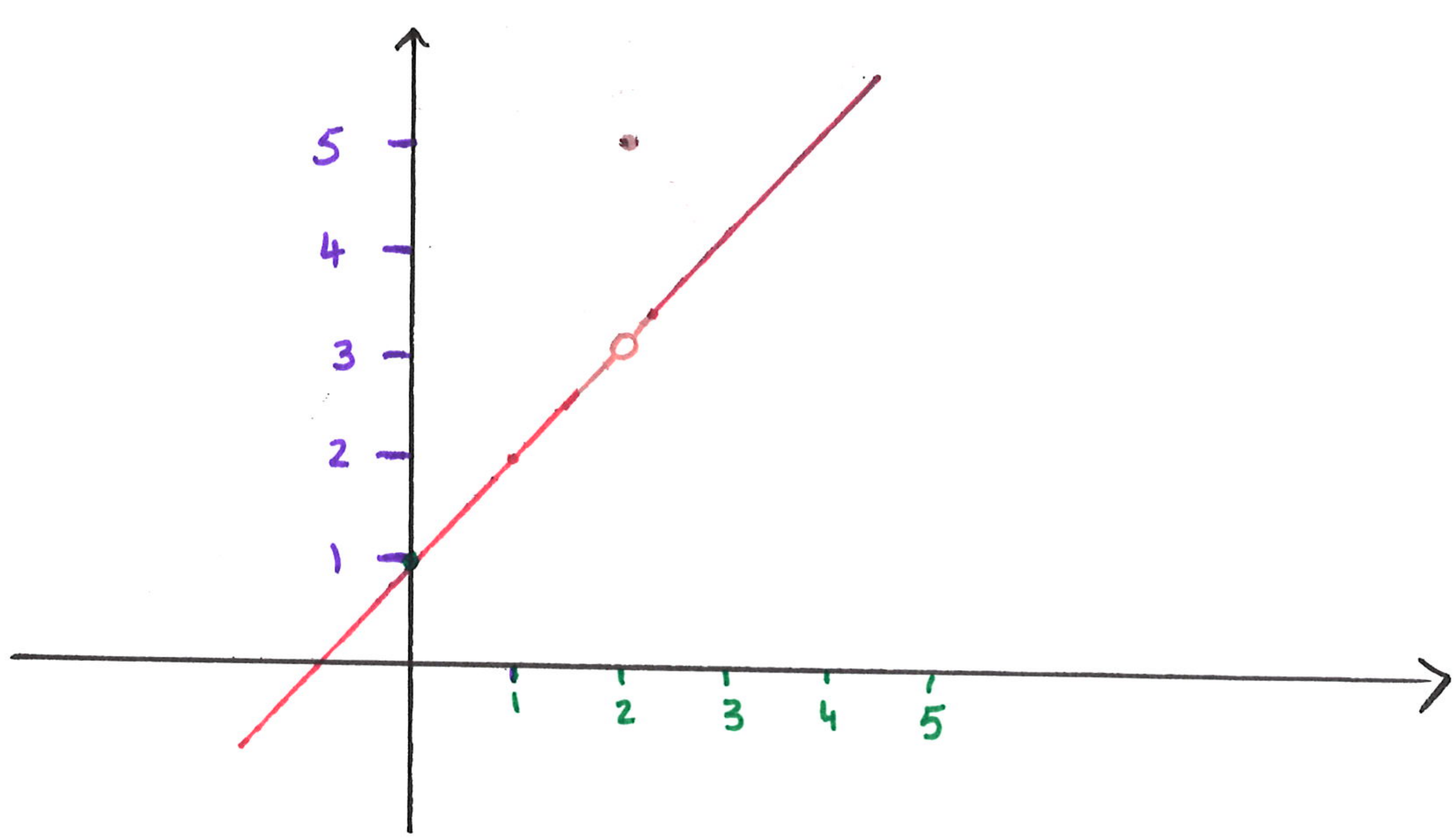
if we can ensure that our output values, $f(x)$, are as close as we want to L, by having our input values, x, sufficiently close to (but not equal to) a.

Picture:

Remark: A table of values like the one in our first example can be useful, but may give the wrong impression (See example $f(x) = \sin\left(\frac{1}{x}\right)$ later).

For now, an accurate graph is the most reliable method we have to find limits.

Example: $g(x) = \begin{cases} x+1 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$



$\lim_{x \to 2} g(x) =$ ______ .

Note:

Remarks:

(1) The value of a function at a plays no role (in general) in the value of $\lim_{x \to a} f(x)$.

(2) If two functions f and g have the same behaviour around a (i.e. $f(x) = g(x)$ for x close to a), then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

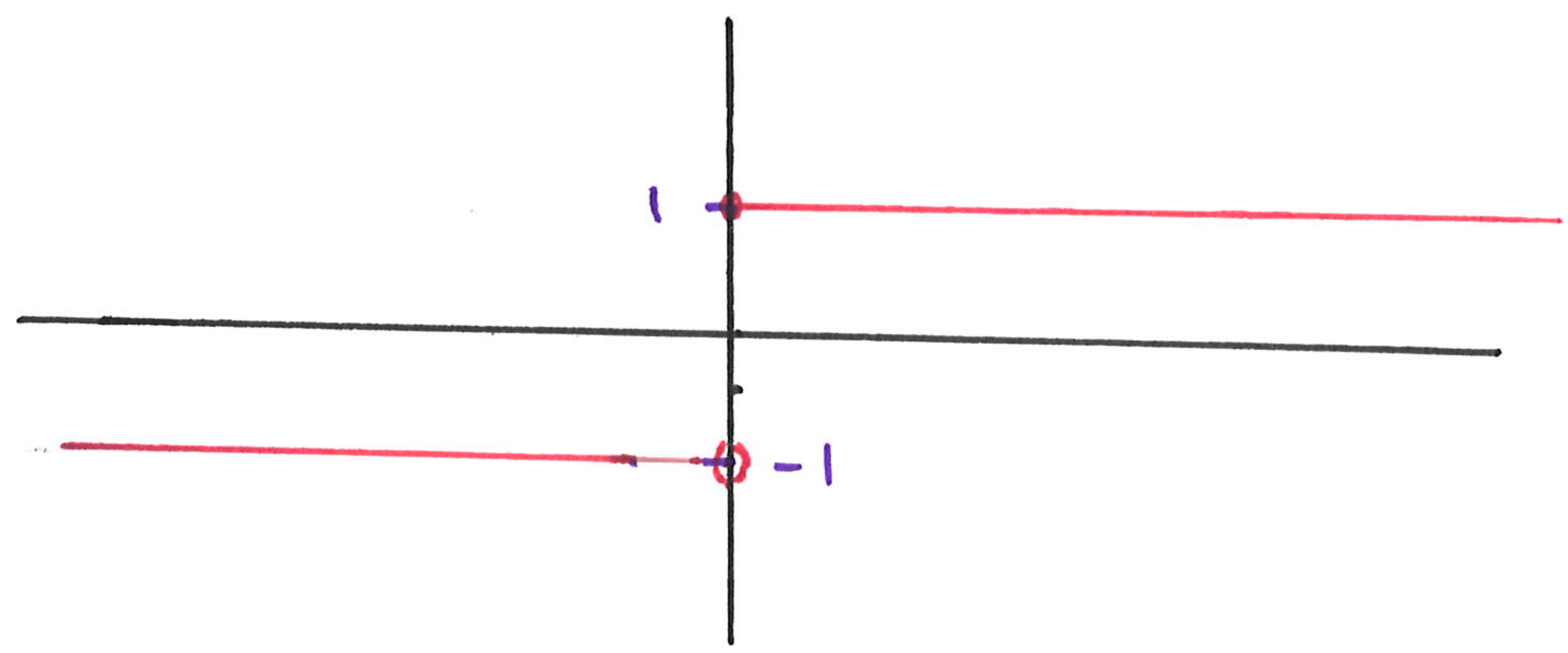
(3) In some cases, our outputs do not settle around any particular L as x approaches a.
i.e. The values $f(x)$ do not have a limit as x approaches a.
We then say $\lim_{x \to a} f(x)$ does not exist.

There are multiple ways this can happen:

Examples:

(1)

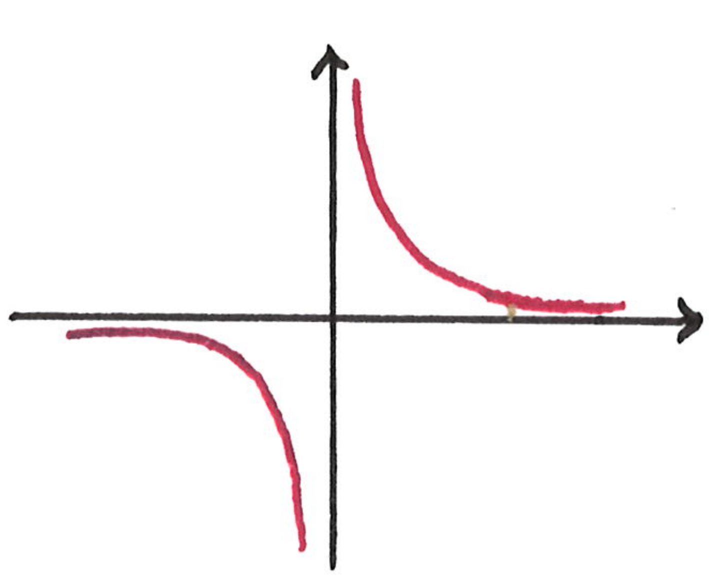
$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Does $\lim_{x \to 0} h(x)$ exist?

(2) $f : (-\infty, 0) \cup (0, \infty) \longrightarrow \mathbb{R}$

$\quad : \quad x \longmapsto 1/x \quad$ (i.e. $f(x) = 1/x$)



Does $\lim_{x \to 0} f(x)$ exist?

Left and Right Hand Limits:

Definitions:

(1) We say the left hand limit of $f(x)$ as x approaches a is equal to L, and write

$$\boxed{\lim_{x \to a^-} f(x) = L}$$

if we can ensure that our output values, $f(x)$, are as close as we want to L, by having our input values, x, sufficiently close to, and less than a. (Approach on left)

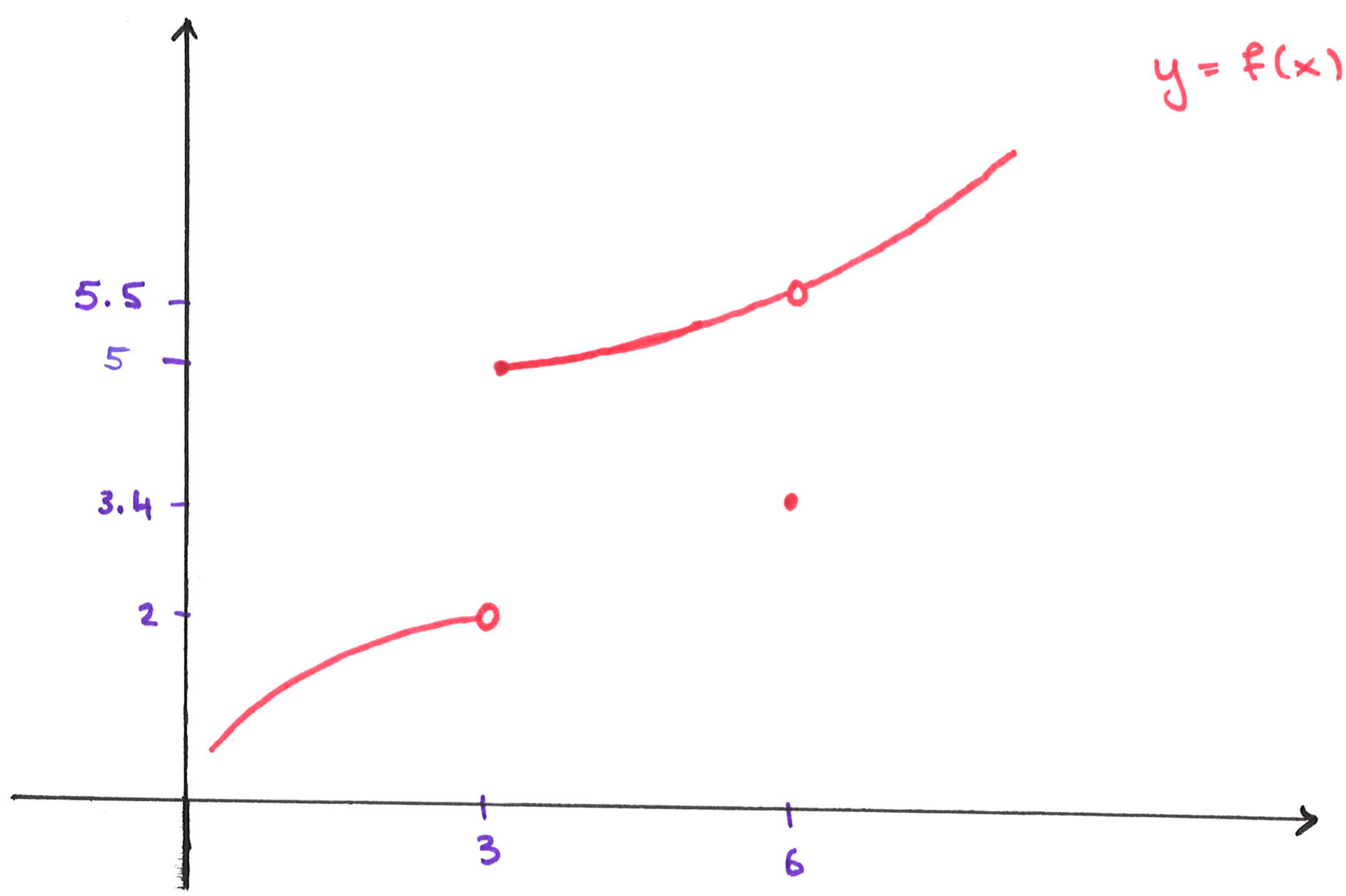
(2) We say the right hand limit of $f(x)$ as x approaches a is equal to L, and write

$$\boxed{\lim_{x \to a^+} f(x) = L}$$

if we can ensure that our output values, $f(x)$, are as close as we want to L, by having our input values, x, sufficiently close to, and greater than a. (Approach on right).

NB: $\boxed{\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)}$

Example: Consider the graph of the function below:



(a) What is $\lim_{x\to 3^-} f(x)$?

(b) What is $\lim_{x\to 3^+} f(x)$?

(c) Does $\lim_{x\to 3} f(x)$ exist?

(d) What is $\lim_{x \to 6^-} f(x)$?

(e) What is $\lim_{x \to 6^+} f(x)$?

(f) Does $\lim_{x \to 6} f(x)$ exist?

Infinite limits:

Definitions:

(1) We write $\boxed{\lim_{x \to a^-} f(x) = \infty}$ if we can ensure the values of $f(x)$ are arbitrarily large by taking x sufficiently close to, and less than a.

Similar definitions are used for:

(2) $\lim_{x \to a^+} f(x) = \infty$

(3) $\lim_{x \to a^-} f(x) = -\infty$

(4) $\lim_{x \to a^+} f(x) = -\infty$

$\Big\}$ arbitrarily large and negative.

Example: Refer back to graph of $f(x) = 1/x$.

Definition: The line $x = a$ is a vertical asymptote to the curve $y = f(x)$ if any of the following are true:

(i) $\lim_{x \to a^-} f(x) = \infty$ (ii) $\lim_{x \to a^+} f(x) = \infty$

(ii) $\lim_{x \to a^-} f(x) = -\infty$ (iii) $\lim_{x \to a^+} f(x) = -\infty$

Examples:

(1) $f(x) = 1/x$ has a vertical asymptote at $x = 0$.

(2) $f(x) = \dfrac{x^2}{1-x}$ has a vertical asymptote at _____.

(3) Does $f(x) = \dfrac{x^2 - 9}{x - 3}$ have a vertical asymptote?

If so, where?

Use this space to answer (3):

In the following example, we will see why a table of values can be misleading when calculating a limit:

Example: $f(x) = \sin\left(\frac{1}{x}\right)$

Let's try to determine $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ with two tables:

x	$\sin\left(\frac{1}{x}\right)$
$\frac{2}{\pi}$	
$\frac{2}{5\pi}$	
$\frac{2}{9\pi}$	
$\frac{2}{13\pi}$	
$\frac{2}{17\pi}$	

x	$\sin\left(\frac{1}{x}\right)$
$\frac{2}{3\pi}$	
$\frac{2}{7\pi}$	
$\frac{2}{11\pi}$	
$\frac{2}{15\pi}$	
$\frac{2}{19\pi}$	

Remark:

Example The graph of $f(x) = \sin(1/x)$ is shown below. If we look at the behavior of the curve as x approaches 0, we see that the graph oscillates between -1 and +1 with increasing frequency. Since the $y-$values on the graph do not approach a unique y-value L as x approaches 0, we have that $\lim_{x\to 0} \sin(1/x)$ does not exist.