§ 2. Tangents and Velocity:

Tangents: The word tangent is derived from the Latin word tangens, which means "touching".

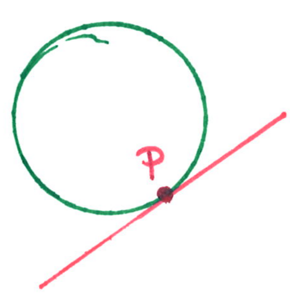
Idea: An initial attempt at a definition of the tangent line to a curve at the point P might be:

"The line which touches P, heading in the same direction as the curve."

Problem: This is a vague definition, as we do not have a precise notion of the "direction of a curve".

Examples:

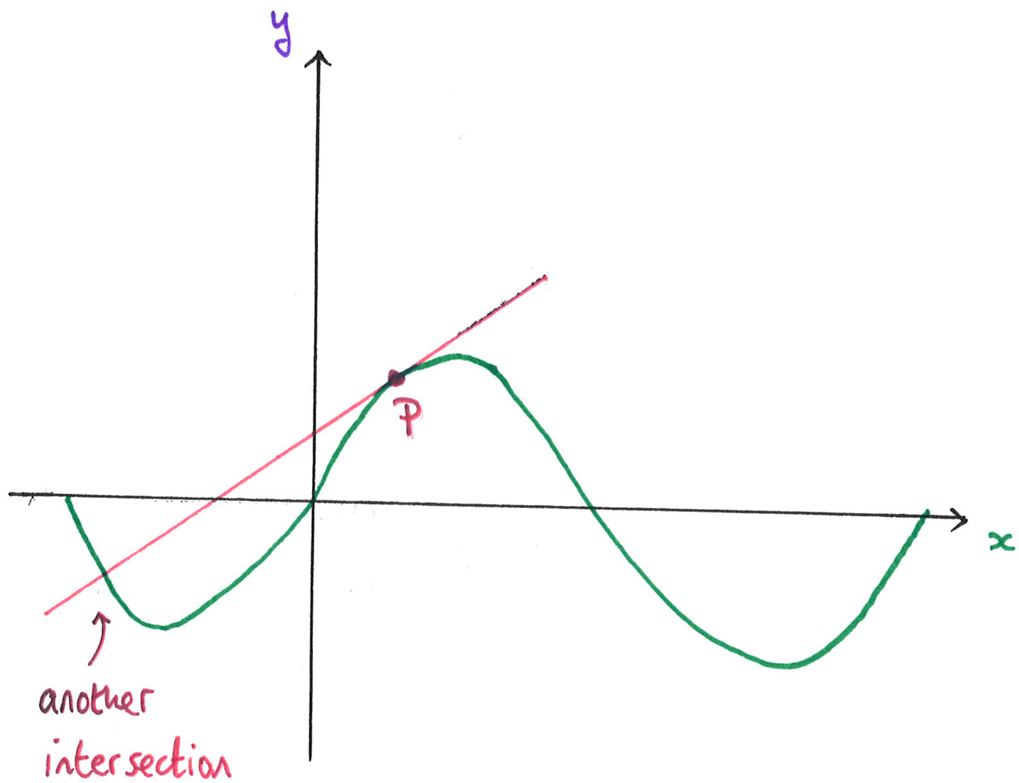
(1) For a circle, we can make a precise definition of the tangent line to the circle at P:



It is the unique line which touches the circle at P and nowhere else.

However, this idea will not work for all curves:

(2)



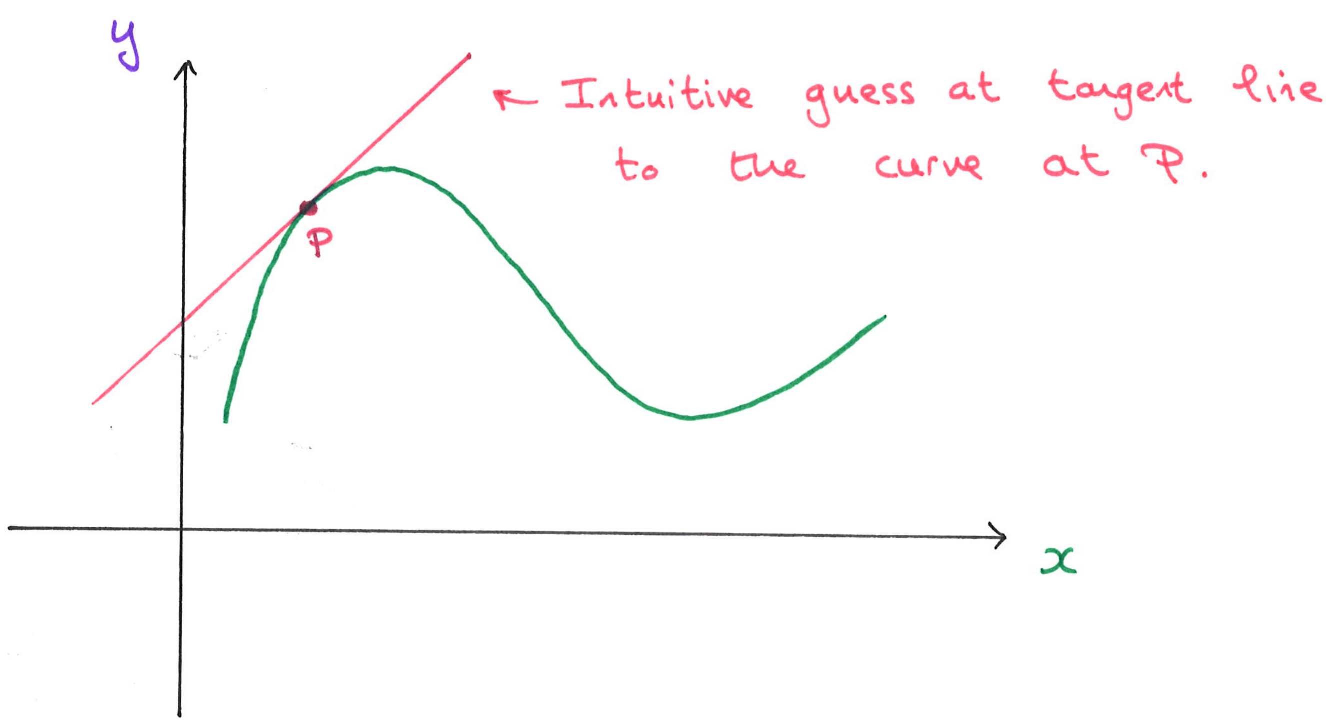
Here we have what we intuitively would like the tangent line at P to be, and we see that it touches the curve more than once.

Goal: Later in the course we will use the concept of a limit to establish a precise notion about what we mean by "the slope | direction of a curve at a point P".

We will then define the tangent line to the curve at P to be the line containing P with the same slope as the curve at P.

However, even without this precise definition, we can use our intuition to make some headway:

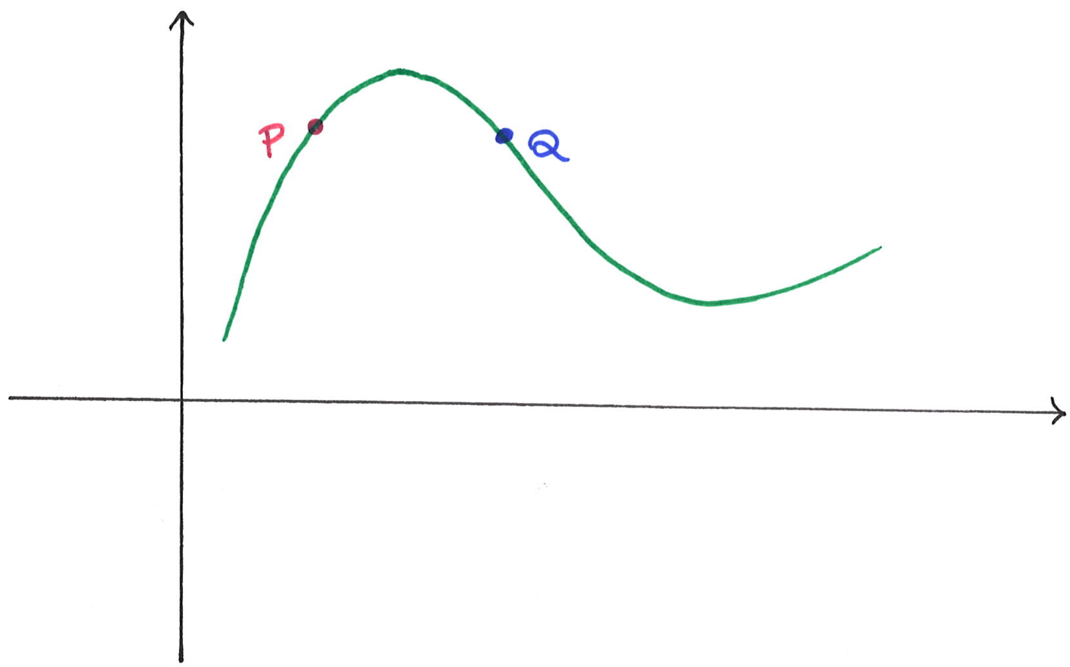
Idea: Say we have a curve in the xy-plane and a point P on the curve.



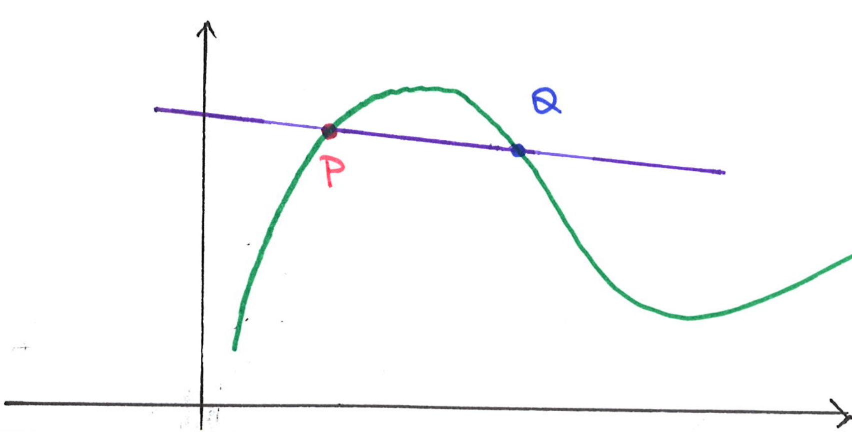
If we want to have a notion of the "direction of the curve at P", it makes sense to think about where the curve is "coming from / going to" around P.

i.e. we should think about points on the curve "near" P:

Consider the point Q on the curve:

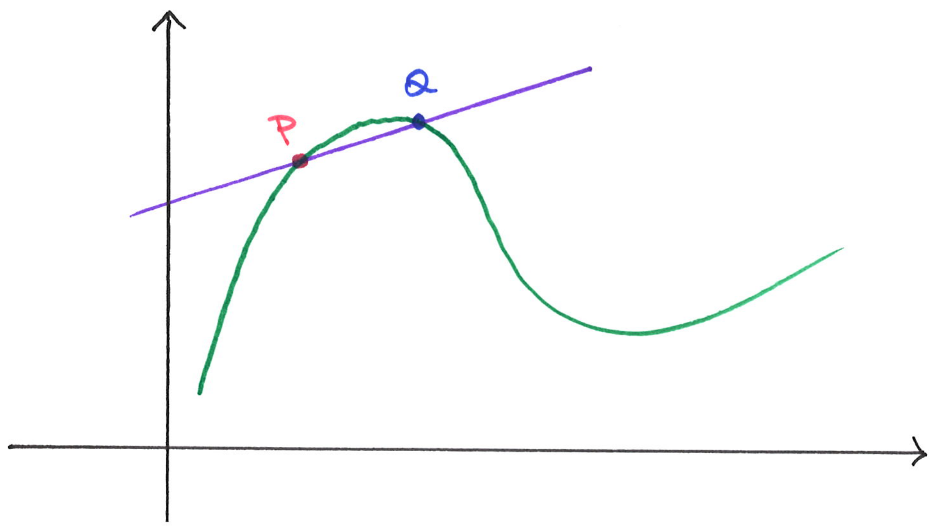


The line that touches P and heads towards Q (secant line) would be:



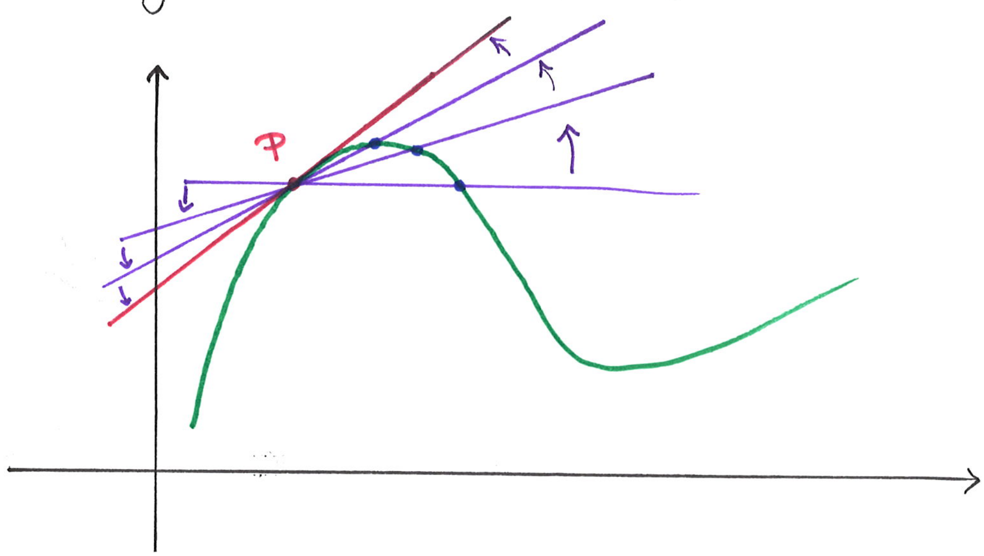
which doesn't look like what we think our tangent should be.

Let's take another Q. This time closer to P:



We can see that this secant line is a "better guess" than our last one.

Now imagine doing this for a closer Q... and then again for an even closer Q...



We can see that, as Q gets closer to P, our guess at the tangent line gets better and better.
In particular, the slopes of our guesses (secant lines) get closer and closer to the slope of our desired tangent line.

Let's look at some concrete examples:

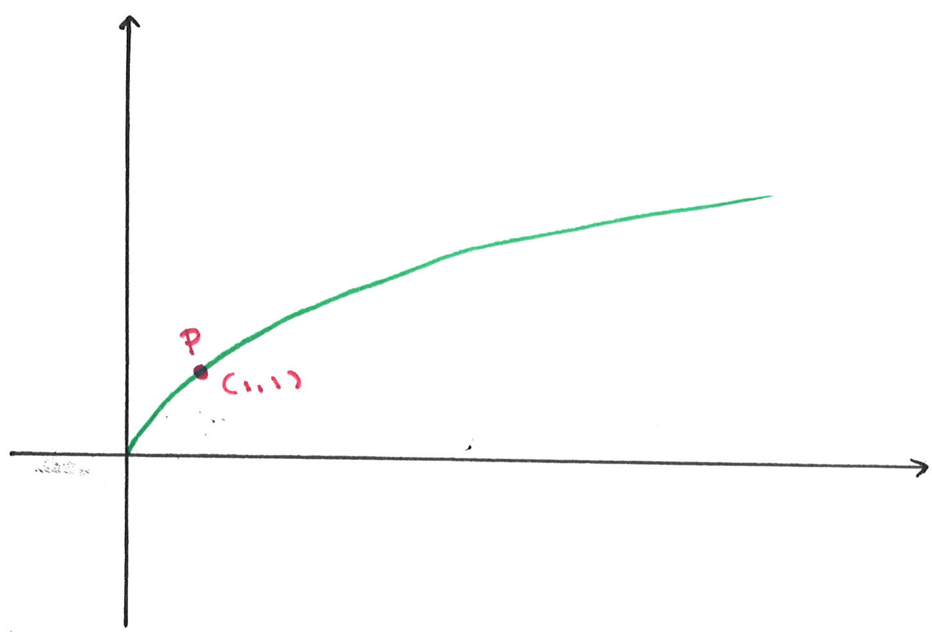
Examples:

(1) Consider the graph of the function

$$f : [0,\infty) \longrightarrow [0,\infty)$$
$$: x \longmapsto \sqrt{x} \quad (\text{i.e. } f(x) = \sqrt{x})$$

(Remark: This can also be referred to as "the curve $y = \sqrt{x}$")

We will attempt to find the tangent line to the curve at the point $P = (1,1)$.



Method:

(1) Create a list of points on the curve which get closer and closer to $(1,1)$.

Recall: All points on the curve have the form ______.

So once we give x-values close to the x-value of P (in our case this is 1), we can use the formula we have for $f(x)$ to make our list of Q's:

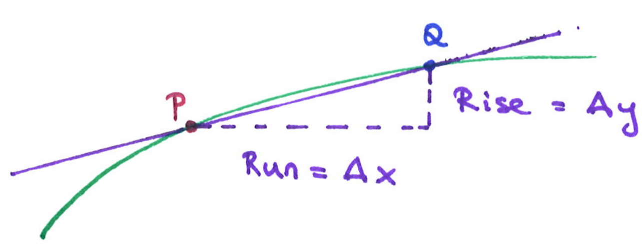
x-value	Q
1.5	$(1.5, \sqrt{1.5})$
1.1	$(1.1, \sqrt{1.1})$
1.01	$(1.01, \sqrt{1.01})$
1.001	$(1.001, \sqrt{1.001})$
1.0000001	$(1.0000001, \sqrt{1.0000001})$

(2) Find the slope of the secant lines connecting P to the various Q's you have chosen (call these slopes m_{PQ}):

change in y

Recall:
$$\text{slope} = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

change in x



x - value	Q	m_{PQ}
1.5	$(1.5, \sqrt{1.5})$	$\frac{\sqrt{1.5}-1}{1.5-1} \approx 0.4495$
1.1	$(1.1, \sqrt{1.1})$	$\frac{\sqrt{1.1}-1}{1.1-1} \approx 0.488$
1.01	$(1.01, \sqrt{1.01})$	$\frac{\sqrt{1.01}-1}{1.01-1} \approx 0.49876$
1.001	$(1.001, \sqrt{1.001})$	$\frac{\sqrt{1.001}-1}{1.001-1} \approx 0.499876$
1.0000001	$(1.0000001, \sqrt{1.0000001})$	$\frac{\sqrt{1.0000001}-1}{1.0000001-1} \approx 0.4999999876$

(3) Try to see what the slopes are getting close to.

In our case, it appears that the slopes are getting close to: ___ .

(4) Write down the equation of the line containing P with this slope:

If instead of Δx for a small change in x, we use h, our points Q would look like;

$$(1+h, \sqrt{1+h^2})$$

So the slopes of our secant lines would look like:

$$M_{PQ} = \frac{\sqrt{1+h^2}-1}{1+h-1} = \frac{\sqrt{1+h^2}-1}{h}$$

We saw as h approaches 0 (i.e. our Q's get closer and closer to P) the values of

$$M_{PQ} = \frac{\sqrt{1+h^2}-1}{h}$$

approach $1/2$.

In the languge of limits we write:

$$\boxed{\lim_{h \to 0} M_{PQ} = \lim_{h \to 0} \frac{\sqrt{1+h^2}-1}{h} = 1/2}$$

Remark: For the graph of a function, we will have that the slope of the tangent line at a point P is:

__________ if the function is increasing around P.

__________ if the function is decreasing aroud P.

Velocity.

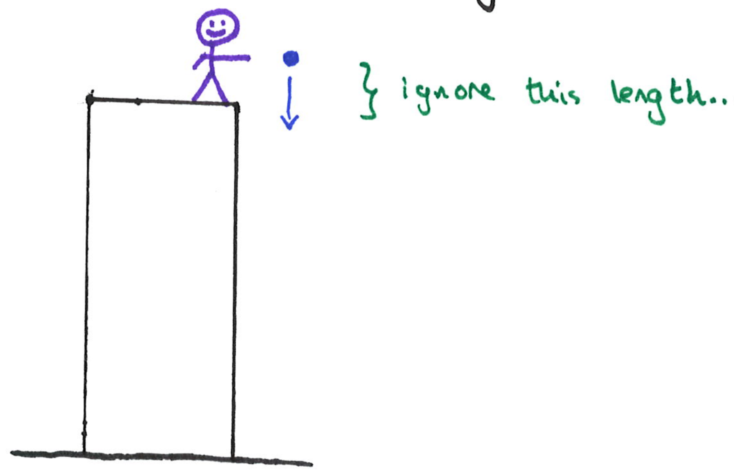
Idea: We can think of the slope of the tangent line at P as the instantaneous rate of change of the curve at P.

This notion is not unnatural to us. We intuitively consider speed to be the instantaneous rate of change of distance covered.

Question: Given a distance function, D, how can we approximate the speed at a given istant?

We shall investigate this with a particular example:

Example: Say I drop a rock off a building which is 490m tall:

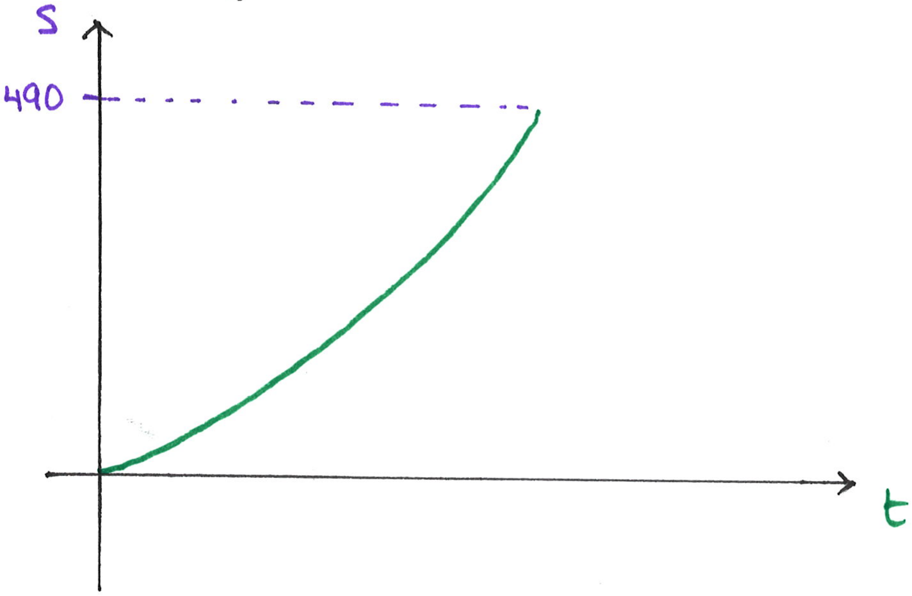


The equation for the distance travelled is (assuming zero initial velocity and no wind or air resistance):

$$\boxed{s(t) = 4.9t^2}$$

Where t is in seconds and s is in meters.

The graph for this function looks like:



(a) How far has the rock travelled after 3 seconds?

(b) How long does it take for the rock to reach the ground?

(c) What was the average speed of the rock on its way to the ground?

(d) Can we think of a way of estimating the rock's speed after 3 seconds?

Is our answer for (c) a good guess?

Hint: Would the average speed of the rock between $t=3$ and $t=4$ be a better guess?

Time interval	Average velocity $= \frac{\Delta s}{\Delta t}$ (in m/s)

Our estimates indicate that the speed at $t = 3$ should be ______ m/s .

Questions:

(1) Can you think of an alternative method to estimate the speed?

(2) Do you notice any similarities between our method for finding the slope of the tangent and the above?