§ I. Functions:

Definition: A set is a collection of distinct objects.

Examples:

(1) $\{1\}$

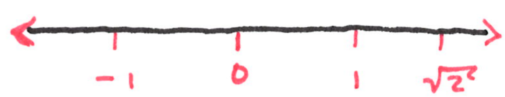
(2) {yellow, blue, red}

(3) Freshman calculus students

(4) $\mathbb{N} = \{1, 2, 3, \ldots\}$ = Positive whole numbers

(5) $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ = Positive and negative whole numbers and 0.

(6) $\mathbb{R}$ = real numbers, (think of the numberline)



(7) Bikes = { , ... }

Notation: If an object, a, is in a set A, we write $a \in A$.

Definition: Given two sets A and B, a **function** f from A to B is a relation (or rule) which assigns, for each element $x \in A$, exactly one element, $f(x) \in B$. We write:

$$f : A \longrightarrow B$$
$$: x \longmapsto f(x)$$

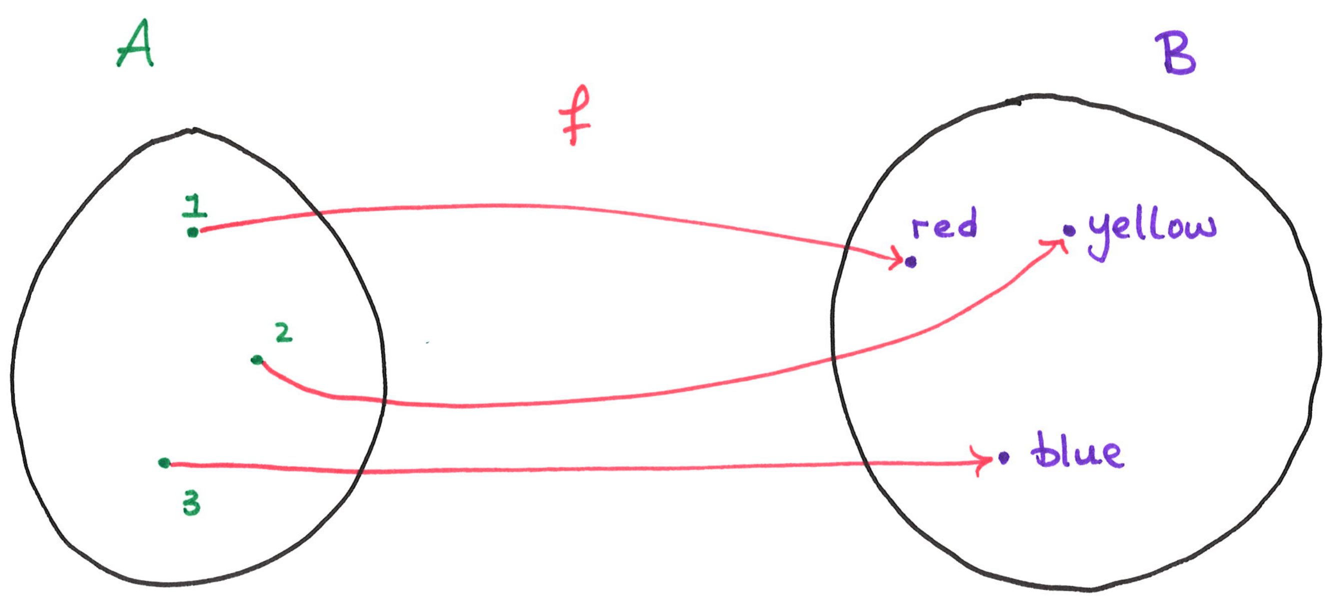
We call A the **domain** of f.
We call B the **codomain** of f.

Examples:

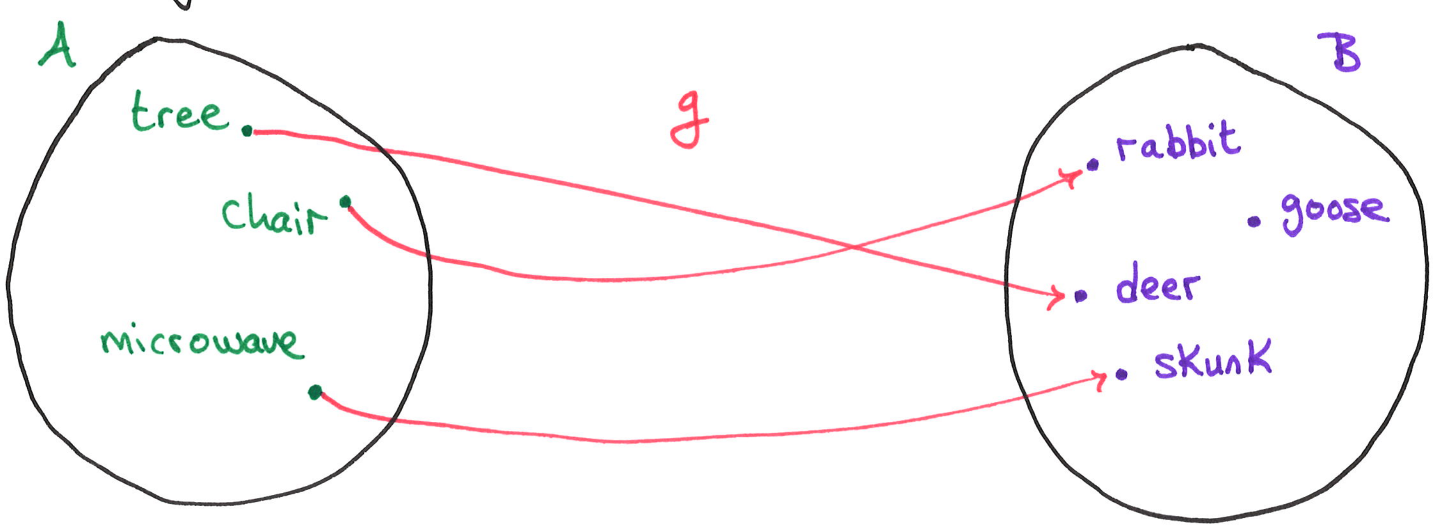
(1) $A = \{1, 2, 3\}$, $B = \{\text{blue}, \text{red}, \text{yellow}\}$

$$f : A \longrightarrow B$$
$$: 1 \longmapsto \text{red} \quad (\text{i.e. } f(1) = \text{red})$$
$$: 2 \longmapsto \text{yellow} \quad (\text{i.e. } f(2) = \text{yellow})$$
$$: 3 \longmapsto \text{blue} \quad (\text{i.e. } f(3) = \text{blue})$$

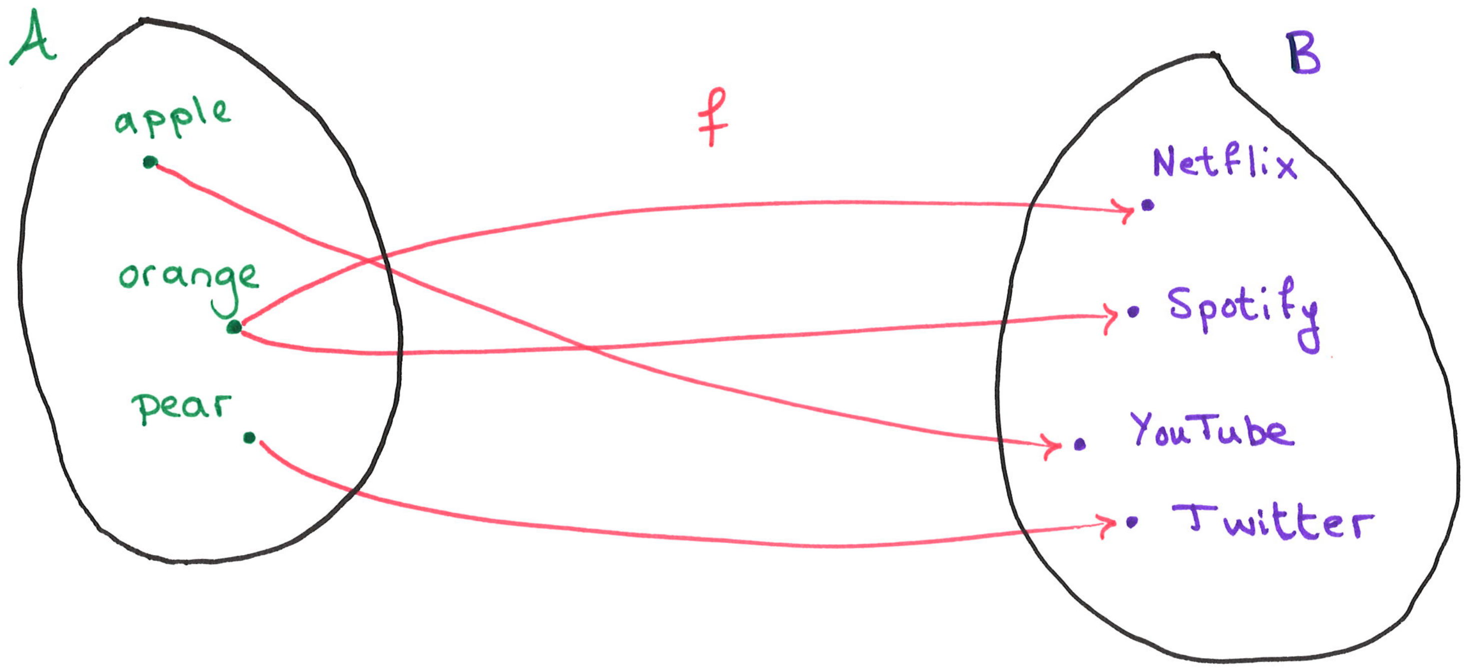
Idea: It can be useful to represent functions with pictures:



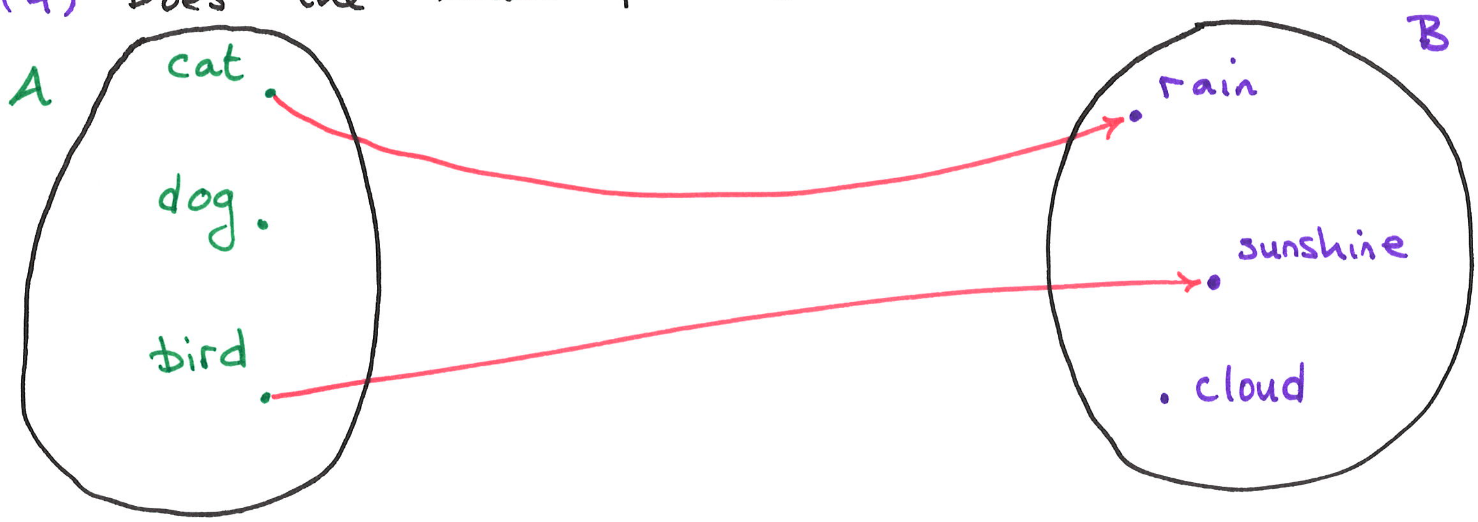
(2) Using the picture below, recover the function:



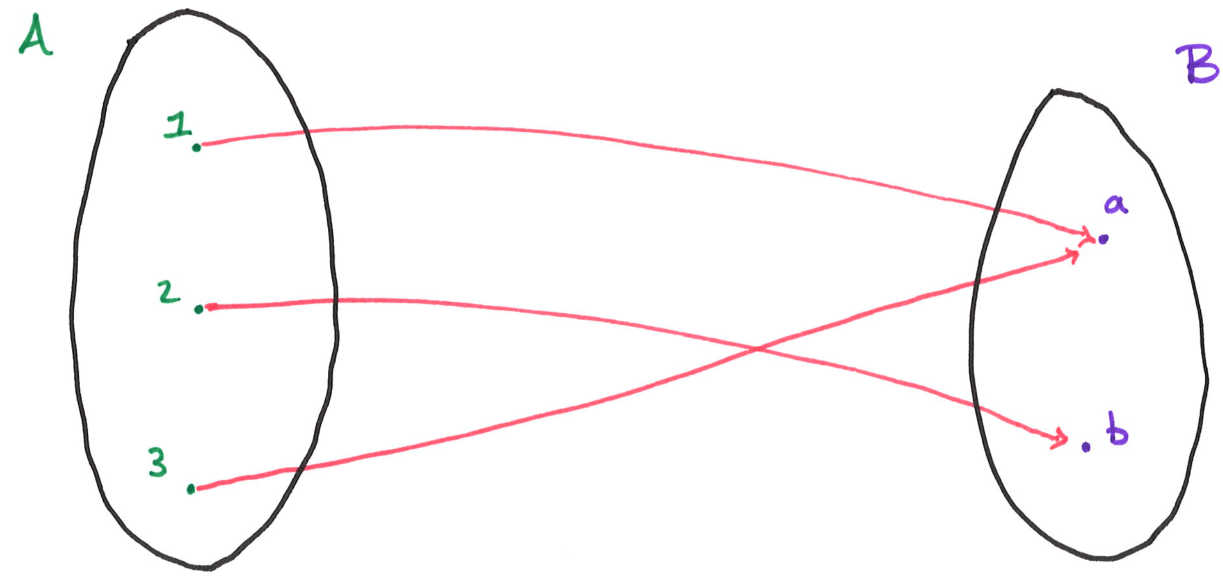
(3) Does the below picture describe a function?



(4) Does the below picture describe a function?

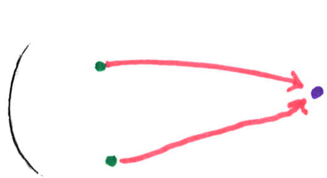


(5) Does the below picture describe a function?



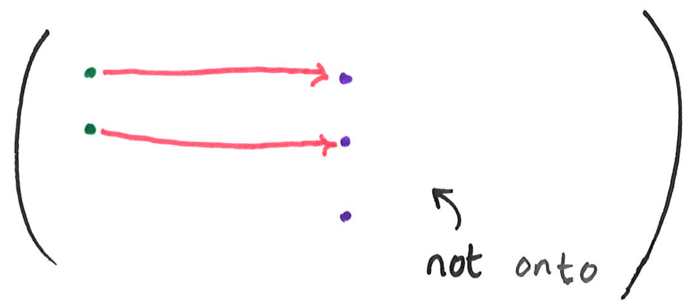
<u>Definitions:</u> Say $f : A \longrightarrow B$ is a function.

(1) If no two elements of A are sent to the same element of B by f, we say f is <u>one-to-one</u> (or <u>injective</u>).

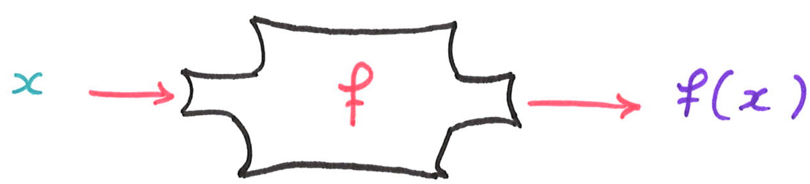


↖ not one-to-one

(2) If for every element of B, there is at least one element of A that is sent to it by f, we say f is <u>onto</u> (or <u>surjective</u>).

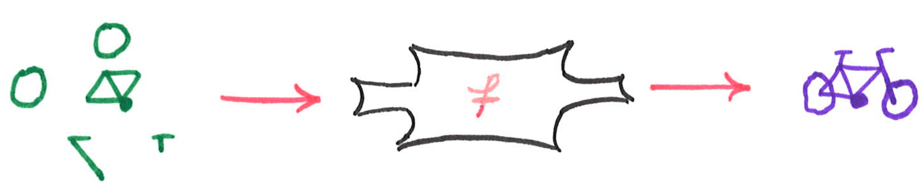


Remark: One way to think about a function is to view it as a machine.
It takes in a certain kind of object (elements of the domain), and spits out another kind of object (elements of the codomain).



Example: (The bike machine)

$$f : \text{Unassembled bikes} \longrightarrow \text{Bikes}$$



Remark: The machine must be consistent.
i.e. every time you put object x into the machine, the machine must produce the same object $f(x)$.

Remark: It is often useful to describe a function using a formula. This is known as expressing a function algebraically.

Examples:

(1) $f : \mathbb{N} = \{1, 2, 3, \ldots\} \longrightarrow \mathbb{N}$

$: n \longmapsto n+1 \quad (\text{i.e. } f(n) = n+1)$

e.g. $f(1) = 1+1 = 2$

$f(16) = 16+1 = 17$

$f(26) = 26+1 = 27$

(2) $f : \mathbb{R} \longrightarrow \mathbb{R}$

$: x \longmapsto 2x+1 \quad (\text{i.e. } f(x) = 2x+1)$

e.g. $f(2) = 2(2)+1 = 5$

$f(-3) = 2(-3)+1 = -5$

$f(1/2) = 2(1/2)+1 = 2$

(3) $f : [0,\infty) \longrightarrow [0,\infty)$

$: \quad x \longmapsto \sqrt{x} \quad$ (i.e. $f(x) = \sqrt{x}$)

e.g. $f(4) = \sqrt{4} = 2$

Alternative view: One very useful way to consider a fuction is as a collection of pairs.
We pair each element of A with the element in B that it is sent to under f.
So each pair is of the form $(x, f(x))$.
We then write these in a set: $\{(x, f(x))\,;\, x \in A\}$.
This set is called the Graph of f, or $\text{Graph}(f)$.

Examples:

(1) $f : \{\text{tree}, \text{coat}, \text{rain}\} \longrightarrow \{\text{yellow}, \text{red}\}$

$: \quad \text{tree} \longmapsto \text{yellow} \quad$ (i.e. $f(\text{tree}) = \text{yellow}$)

$: \quad \text{coat} \longmapsto \text{yellow} \quad$ (i.e. $f(\text{coat}) = \text{yellow}$)

$: \quad \text{rain} \longmapsto \text{red} \quad$ (i.e. $f(\text{rain}) = \text{red}$)

Then, $\text{Graph}(f) = \{(\text{tree}, \text{yellow}), (\text{coat}, \text{yellow}), (\text{rain}, \text{red})\}$

(2) $g : \mathbb{R} \longrightarrow \mathbb{R}$

$: x \longmapsto 2x+1$ (i.e. $g(x) = 2x+1$)

Then, $\text{Graph}(g) = \{(x, g(x)) ; x \in \mathbb{R}\}$

$= \{(x, 2x+1) ; x \in \mathbb{R}\}$

NB

Remark: Another extremely useful way of representing a function (probably the most useful for this course - along with algebraically), is to represent the Graph of the function as a picture.

This is creatively called "graphing the function".

It is especially useful with dealing with functions whose domain | codomain is a subset of $\mathbb{R}$.

These make up the bulk of the functions we will see in this course.

Examples:

(1) $f : \{1,2,3\} \longrightarrow \{1,2,3\}$

$: 1 \longmapsto 2$ (i.e. $f(1) = 2$)

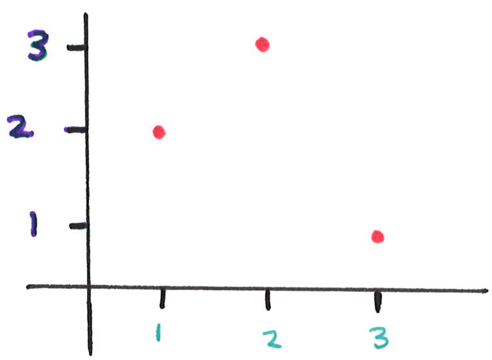
$: 2 \longmapsto 3$ (i.e $f(2) = 3$)

$: 3 \longmapsto 1$ (i.e. $f(3) = 1$)

So $\text{Graph}(f) = \{(1,2),(2,3),(3,1)\}$

As a picture:

Codomain axis is usually vertical



Domain axis is usually horizontal

(2) $f : \mathbb{R} \longrightarrow \mathbb{R}$

$: x \longmapsto x$ (i.e. $f(x) = x$)

e.g. $f(0) = 0 \Rightarrow (0,0) \in \text{Graph}(f)$

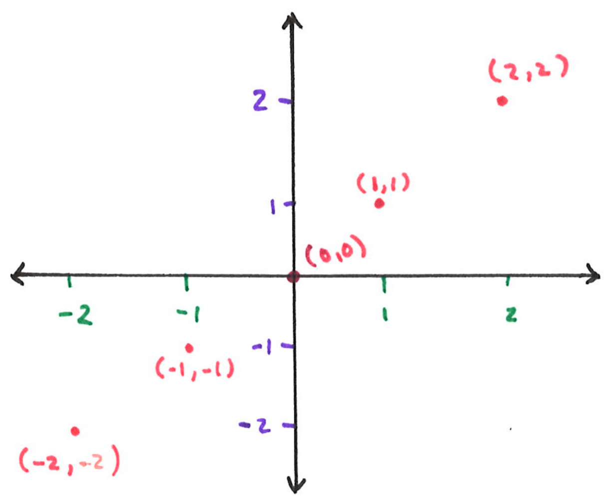
$f(3) = 3 \Rightarrow (3,3) \in \text{Graph}(f)$

$f(\sqrt{2}) = \sqrt{2} \Rightarrow (\sqrt{2}, \sqrt{2}) \in \text{Graph}(f)$

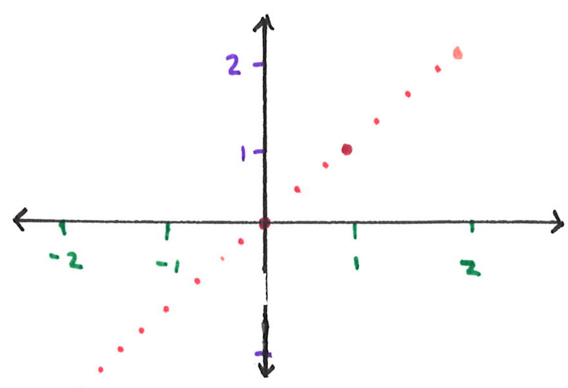
So, $\mathrm{Graph}(f) = \{ (x, x) \,;\, x \in \mathbb{R} \}$

As a picture:

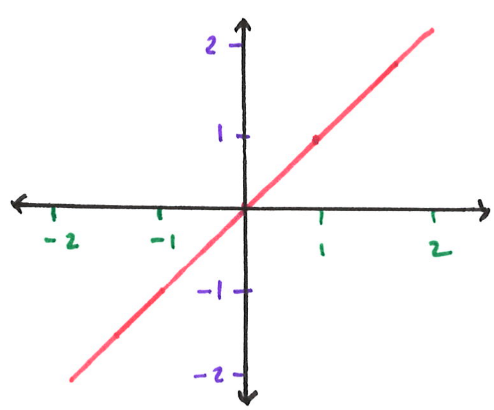
- Let's just plot a few points to begin with:



- Keep adding points...



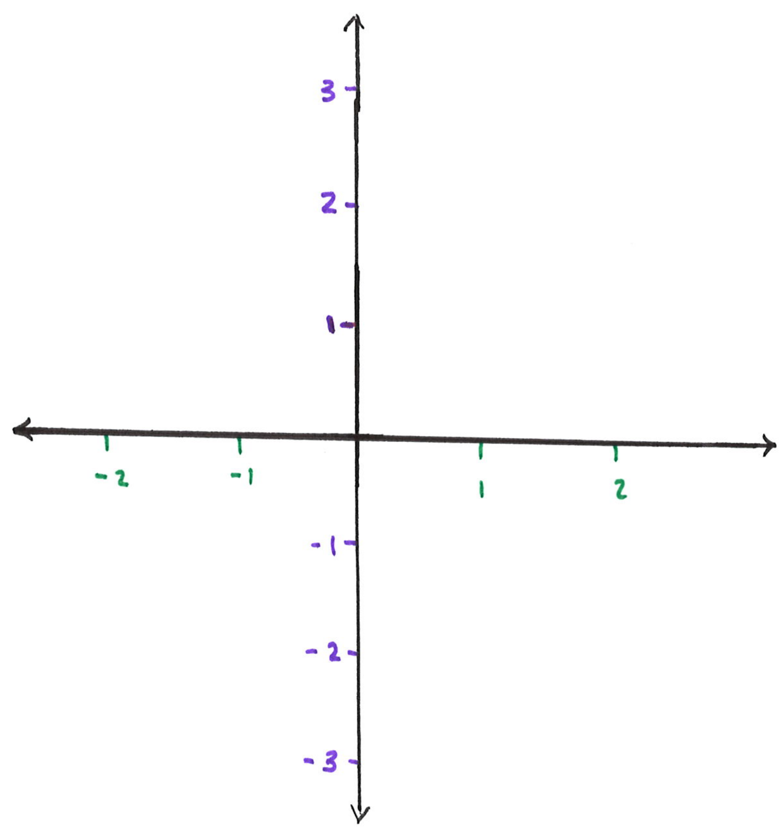
- Until "eventually"...



Remark: The previous is a case where we have "so many" points to plot, that the image we get from graphing f is a solid line | curve, rather than a collection of isolated dots. This will be the case for almost every function we graph in this course.

(3) Graph the following function:

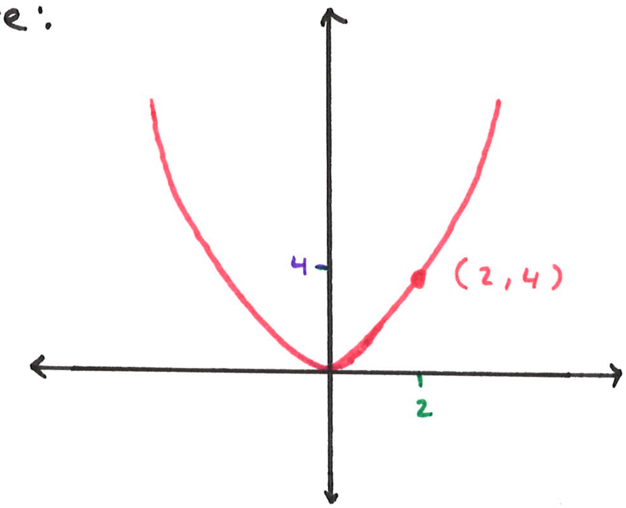
$$g: \mathbb{R} \longrightarrow \mathbb{R}$$
$$: x \longmapsto 2x+1 \quad (\text{i.e. } g(x) = 2x+1)$$



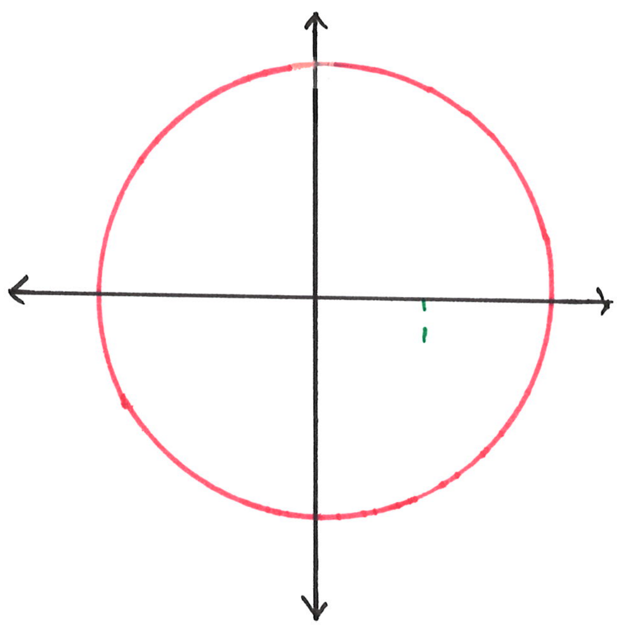
(4) $f : \mathbb{R} \longrightarrow \mathbb{R}$

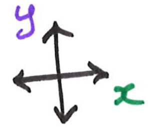
$: x \longmapsto x^2$ (i.e. $f(x) = x^2$)

As a picture: (... roughly)



(5) Can the below picture be from the graph of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$?



Remark: This picture: 

is often called "the xy plane".

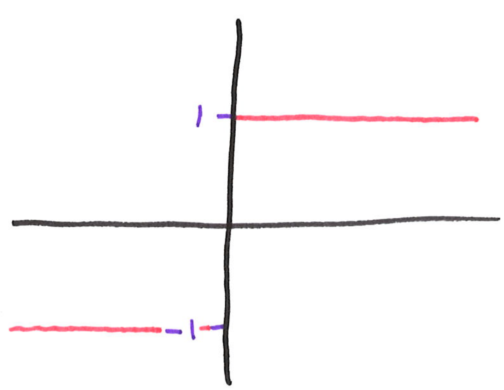
The previous example motivates the following:

The Vertical Line Test: A curve in the xy-plane represents a function $f : \mathbb{R} \to \mathbb{R}$ if and only if no vertical line intersects the curve more than once.

Definitions:

(1) A function is said to be defined piecewise if it is defined using different formulas in different parts of it's domain.

e.g. $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$



(2) A function $f : \mathbb{R} \to \mathbb{R}$ is called even if $f(x) = f(-x)$ for all $x \in \mathbb{R}$.

e.g. $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$

$$\left(f(x) = x^2 = (-x)^2 = f(-x) \right)$$

(3) A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$.

e.g. $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x$

$$\left(f(x) = -(-x) = -f(-x) \right)$$

Remark:

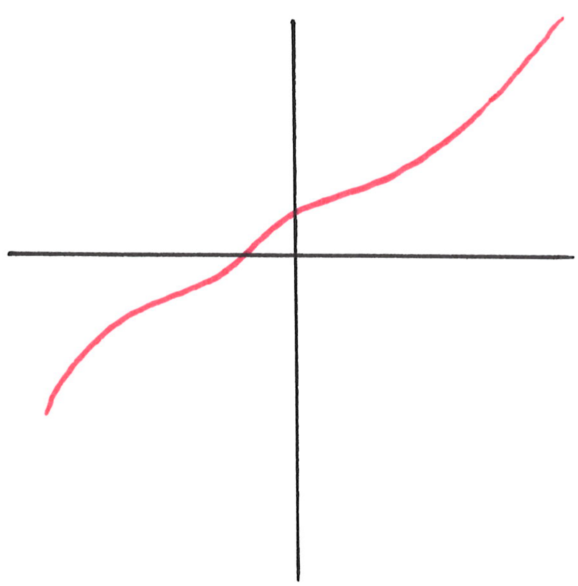
(1) The graph of an even fuction is symmetric about the y-axis:

(2) The graph of an odd function is symmetric about the origin:

(4) A function $f : \mathbb{R} \to \mathbb{R}$ is called <u>increasing</u> on an interval I if

$f(x_1) < f(x_2)$ whenever $x_1 < x_2$, for $x_1, x_2 \in I$.

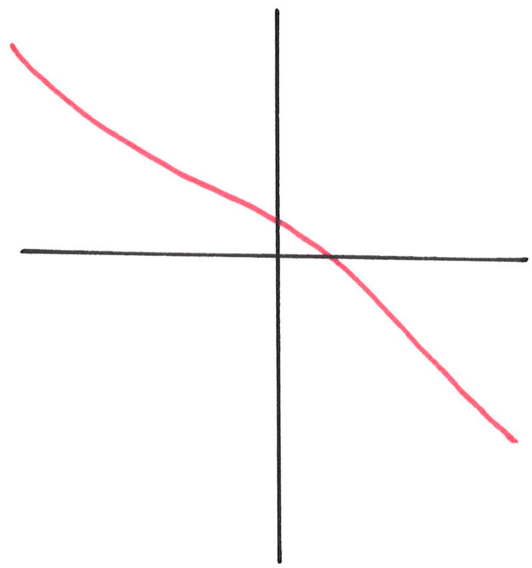
e.g.



(5) A function $f : \mathbb{R} \to \mathbb{R}$ is called decreasing on an interval I if

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$, $x_1, x_2 \in I$.

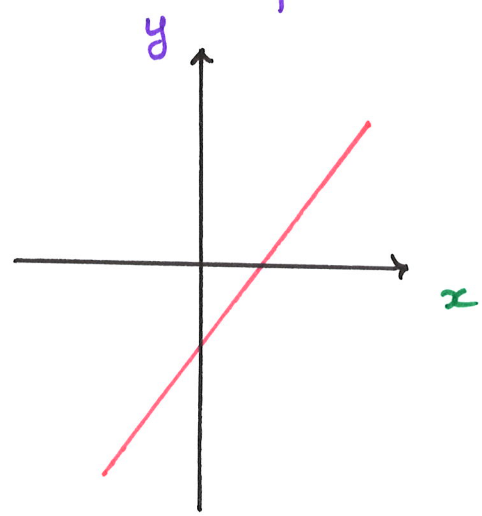
e.g.



A Catalog of Essential Functions:

(1) We say that a function $f: \mathbb{R} \to \mathbb{R}$ is linear if, when we graph f, the resulting image is a line. Algebraically, these functions take the form:

$$f(x) = \underbrace{m}_{\text{slope}} x + \underbrace{b}_{\text{'y-intercept'}}$$



Uses: Break even analysis for business, etc.

(2) A function $P: \mathbb{R} \to \mathbb{R}$ is called a polynomial if, algebraically, it has the following form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a positive integer ($n \in \mathbb{N}$) and $a_0, a_1, \ldots, a_n$ are constants called the coefficients of the polynomial.

If $a_n \neq 0$, then the degree of the polynomial is n.

e.g. $P : \mathbb{R} \to \mathbb{R}$

$: x \mapsto x^7 + x^2 + 2x$

Then P is a polynomial of degree 7.

(3) We say a function $f : \mathbb{R} \to \mathbb{R}$ is rational if, algebraically, it is given as a quotient of polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$ Polynomials

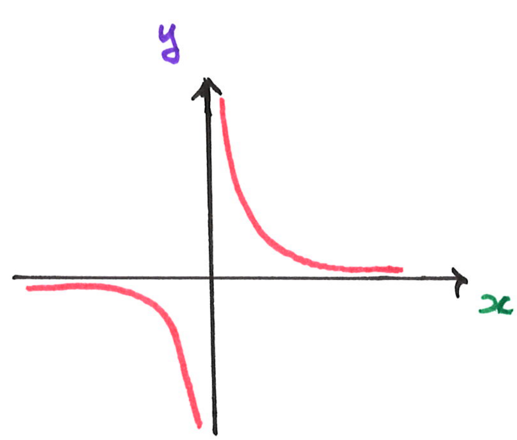
e.g. $$f(x) = \frac{x^3 + 3}{x^2 + x + 1}$$

(4) A power function is a function which algebraically has the form

$$f(x) = x^a$$

where a is a constant.

e.g. $f(x) = \frac{1}{x} = x^{-1}$

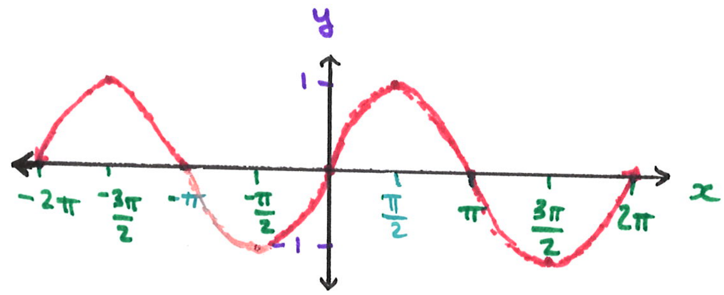


(5) A function f is called an <u>algebraic</u> function if it's algebraic representation can be given in terms of addition, subtraction, multiplication, division, taking roots, etc.

e.g. $f(x) = \sqrt{x^2+1} + \dfrac{x^4+1}{x+\sqrt{x}}$

(6) <u>Trigonometric Functions</u>:

• <u>$f(x) = \sin(x)$:</u>

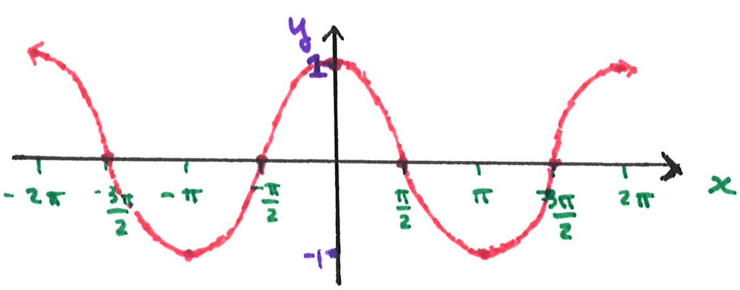


<u>Remarks:</u>

(1) $-1 \leq \sin(x) \leq 1$, for all $x \in \mathbb{R}$

(2) $\sin(x+2\pi) = \sin(x)$, for all $x \in \mathbb{R}$ (periodic)

• <u>$f(x) = \cos(x)$:</u>



<u>Remarks:</u>

(1) $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$

(2) $\cos(x+2\pi) = \cos(x)$, for all $x \in \mathbb{R}$ (periodic)

- $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$:

Remarks:

(1) $\tan(x)$ is undefined whenever $\cos(x) = 0$ which happens for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

(2) $\tan(x+\pi) = \tan(x)$, for all x where $\tan(x)$ is defined.

- $f(x) = \csc(x) = \dfrac{1}{\sin(x)}$

- $f(x) = \sec(x) = \dfrac{1}{\cos(x)}$

- $f(x) = \cot(x) = \dfrac{1}{\tan(x)}$

Remark: Please refer to textbook (or wikipedia) for the graphs \ domains of these functions.

(7) A function $f: \mathbb{R} \to \mathbb{R}$ is said to be an exponential function if, algebraically, it has the form $f(x) = b^x$, where b is a positive constant.

Uses: Modelling natural growth | decay.

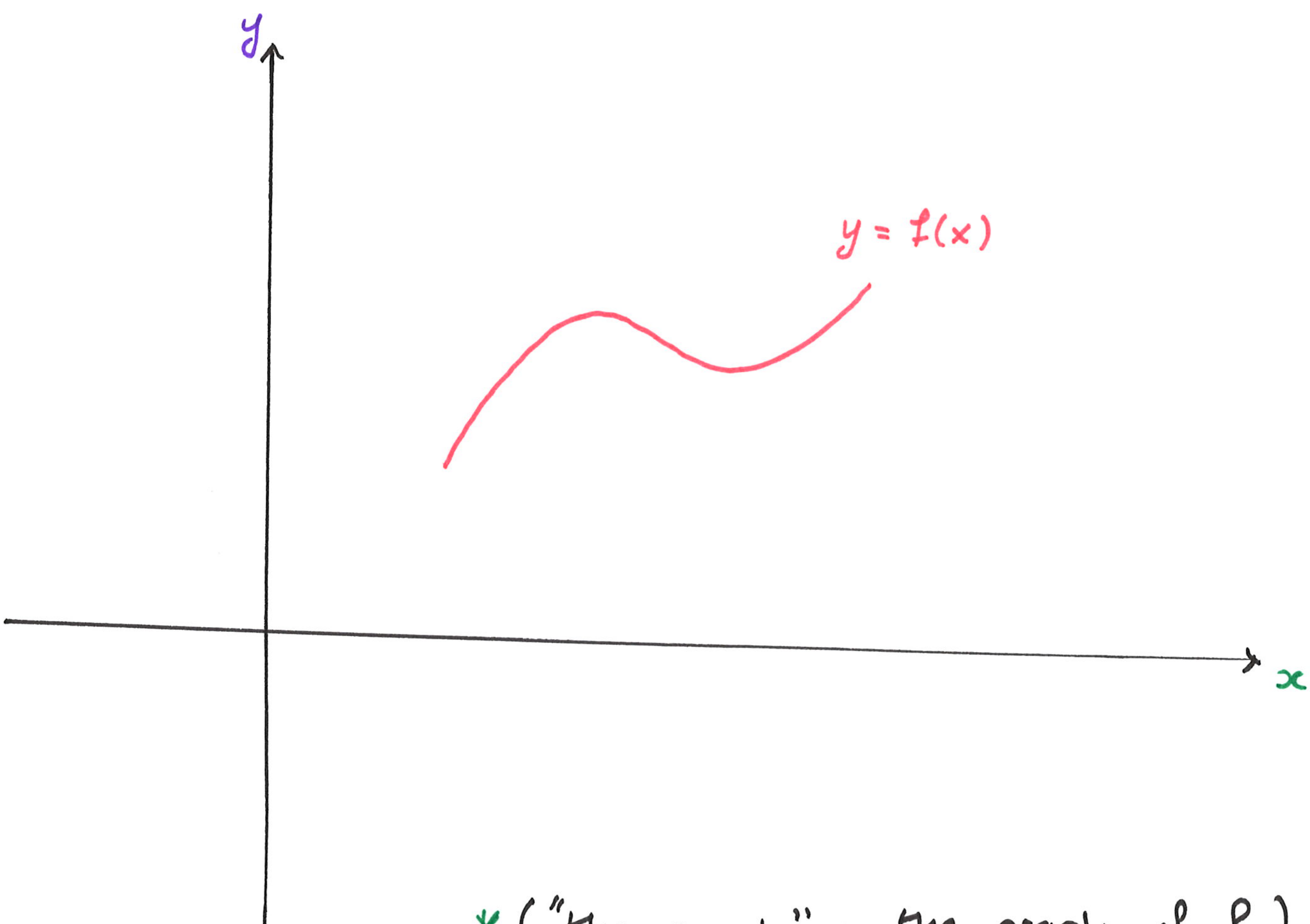
(8) A function $f: (0, \infty) \to \mathbb{R}$ is said to be a logarithmic function if, algebraically, it has the form $f(x) = \log_b(x)$, where b is a positive constant known as the base.

Uses: Modelling natural growth | decay.

Transformations of Functions: Given a function f:

Vertical and Horizontal Shifts: Suppose $c > 0$. To obtain the graph of functions which are given algebraically by:

- $y = f(x) + c$, we shift the graph* c units upwards.
- $y = f(x) - c$, we shift the graph c units downwards.
- $y = f(x-c)$, we shift the graph c units right.
- $y = f(x+c)$, we shift the graph c units left.



* ("the graph" = the graph of f)

Vertical and Horizontal Stretching and Reflecting:

Suppose $c > 1$. To obtain the graph of functions which are given algebraically by:

- $y = cf(x)$, stretch the graph of f vertically by a factor of c.
- $y = (1/c)f(x)$, shrink the graph of f vertically by a factor of c.
- $y = f(cx)$, shrink the graph of f horizontally by a factor of c.
- $y = f(x/c)$, stretch the graph of f horizontally by a factor of c.
- $y = -f(x)$, reflect the graph of f about the x-axis.
- $y = f(-x)$, reflect the graph of f about the y-axis.

* Please see page 37 of textbook for pictures. *

Combinations of functions:

There are many ways we can combine functions. For functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we may define :

(1)
$$\left.\begin{aligned} f+g &: \mathbb{R} \longrightarrow \mathbb{R} \\ &: x \longmapsto f(x)+g(x) \\ \text{i.e. } (f+g)(x) &:= f(x)+g(x) \end{aligned}\right\} \text{Adding functions}$$

(2)
$$\left.\begin{aligned} f-g &: \mathbb{R} \longrightarrow \mathbb{R} \\ &: x \longmapsto f(x)-g(x) \\ \text{i.e. } (f-g)(x) &:= f(x)-g(x) \end{aligned}\right\} \text{Subtracting functions}$$

(3)
$$\left.\begin{aligned} fg &: \mathbb{R} \longrightarrow \mathbb{R} \\ &: x \longmapsto f(x)g(x) \\ \text{i.e. } (fg)(x) &:= f(x)g(x) \end{aligned}\right\} \text{Multiplying functions}$$

Remark: Division is more complicated. We must avoid potentially dividing by 0 :

(4) $$f/g : \mathbb{R} \setminus \{x \,;\, g(x) = 0\} \longrightarrow \mathbb{R}$$
$$: \quad x \longmapsto \frac{f(x)}{g(x)}$$

i.e. $$\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$$

$\Bigg\}$ Dividing functions

Remark: Removing $\{x \,;\, g(x) = 0\}$ from the domain prevents the possibility of dividing by 0.

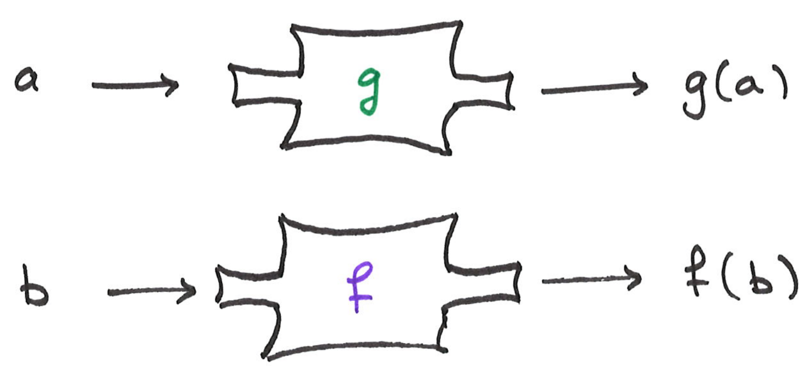
Compositions of functions:

Definition: Say $g : A \to B$ and $f : B \to C$ are functions. We define the composition of f and g to be the function:

$$\begin{aligned} f \circ g &: A \longrightarrow C \\ &: x \longmapsto f(g(x)) \end{aligned}$$

i.e. $(f \circ g)(x) := f(g(x))$

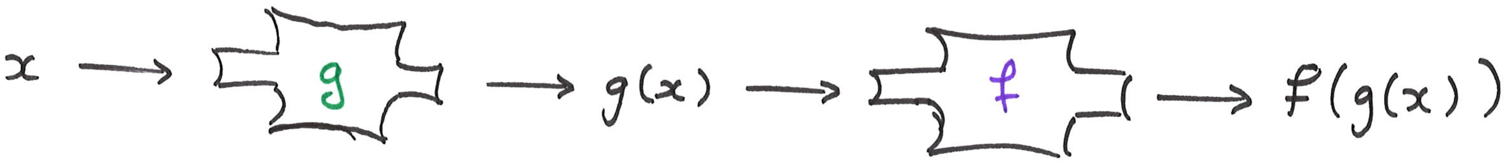
Idea: Recall our machine analogy:



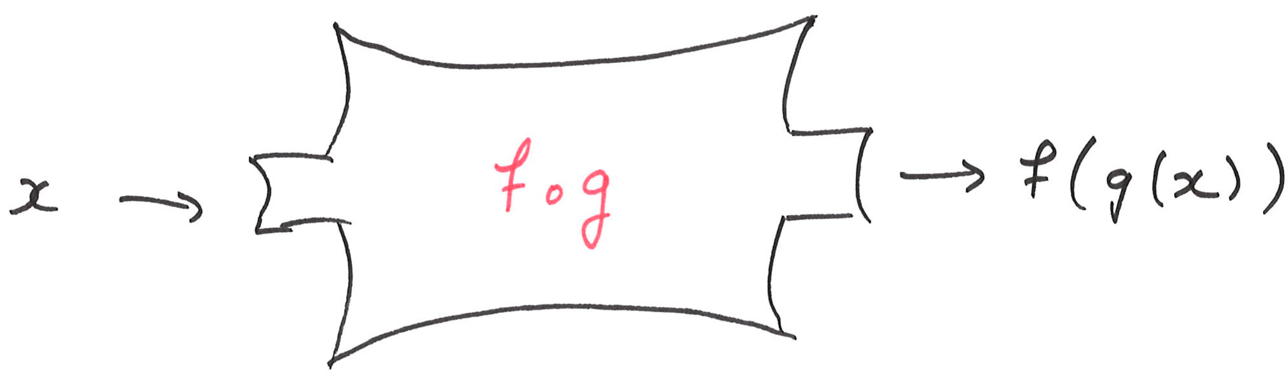
g takes in objects from A and outputs objects in B.

f takes in objects from B and outputs objects in C.

Think of $f \circ g$ as putting these machines beside each other in an assembly line:

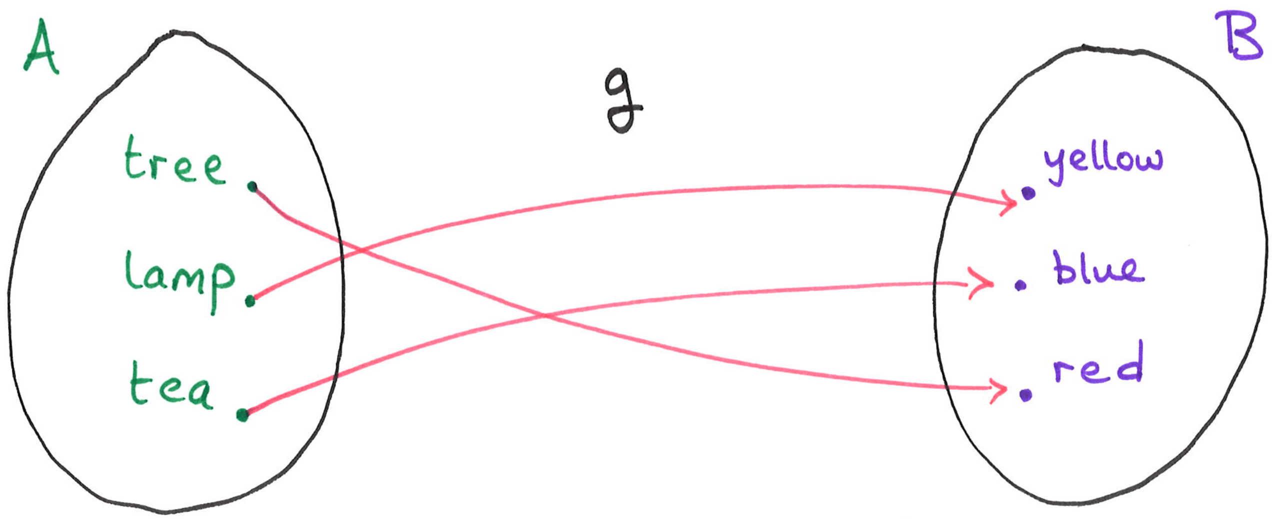


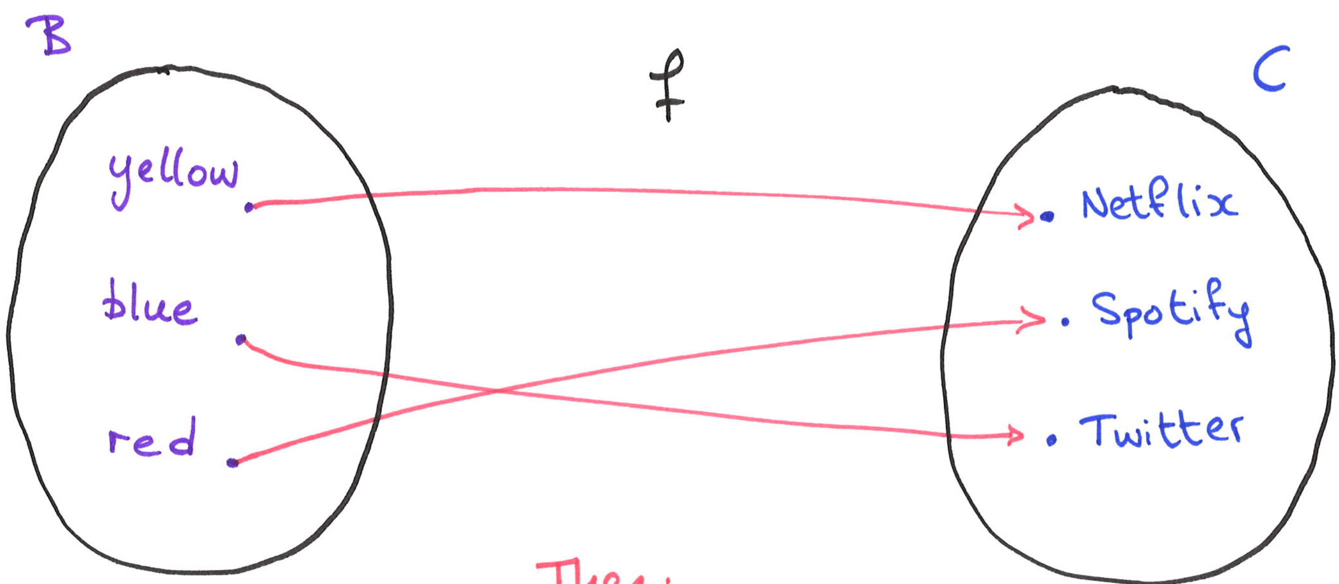
Combine these to make one big machine



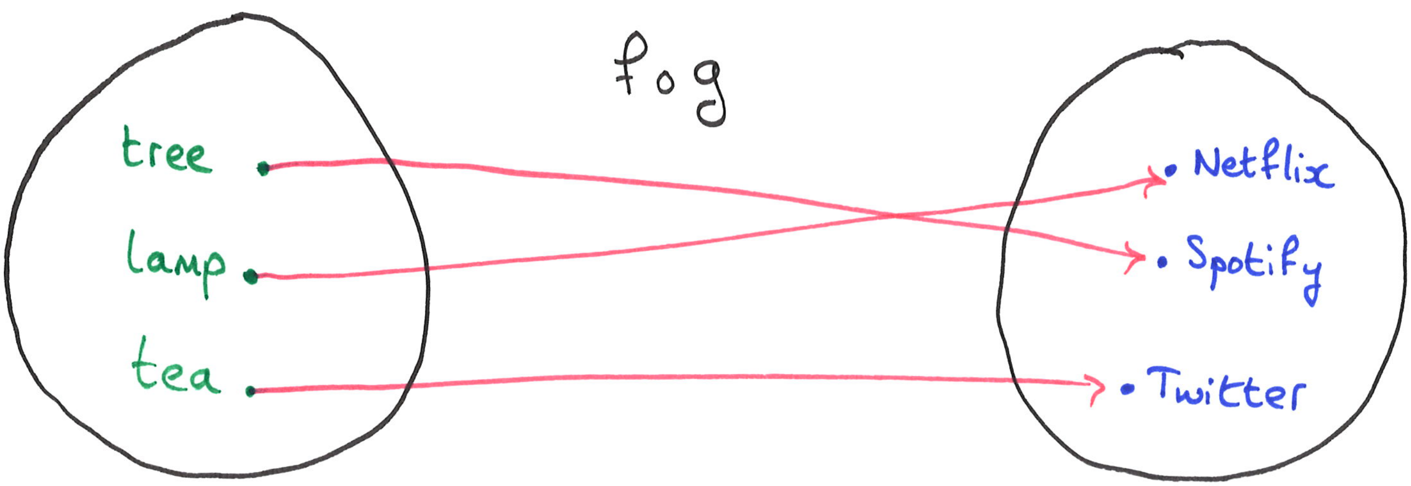
Examples:

(1) Consider the two functions represented in pictures below:



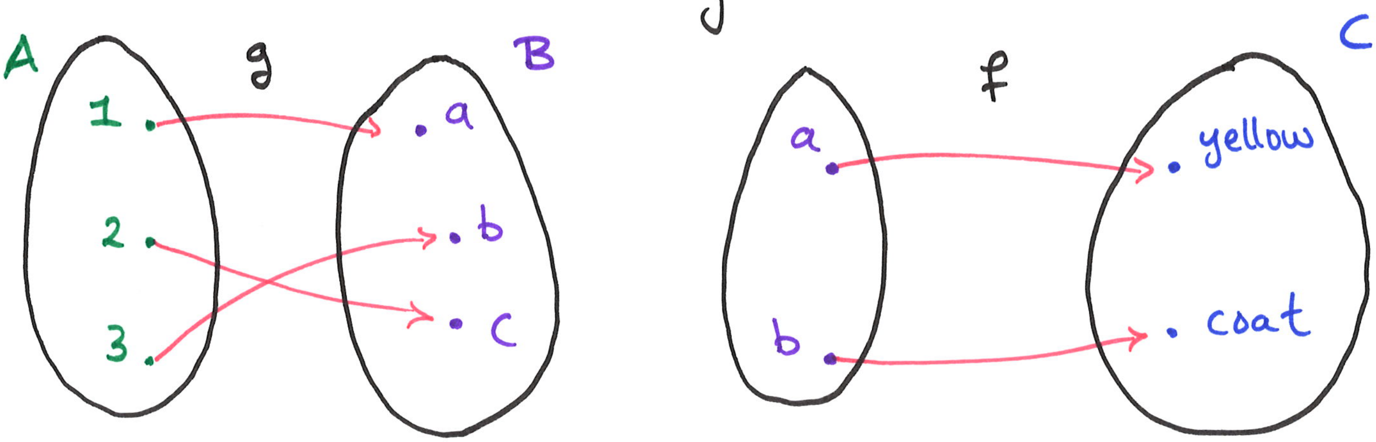


Then:



$$\begin{tikzcd} \Big(\text{tree} \arrow[r, mapsto] \arrow[rrrr, bend right, "f \circ g"'] & \boxed{g} \arrow[r] & \text{red} \arrow[r] & \boxed{f} \arrow[r] & \text{spotify} \Big) \end{tikzcd}$$

(2) Consider the following functions:



Can we compose these?

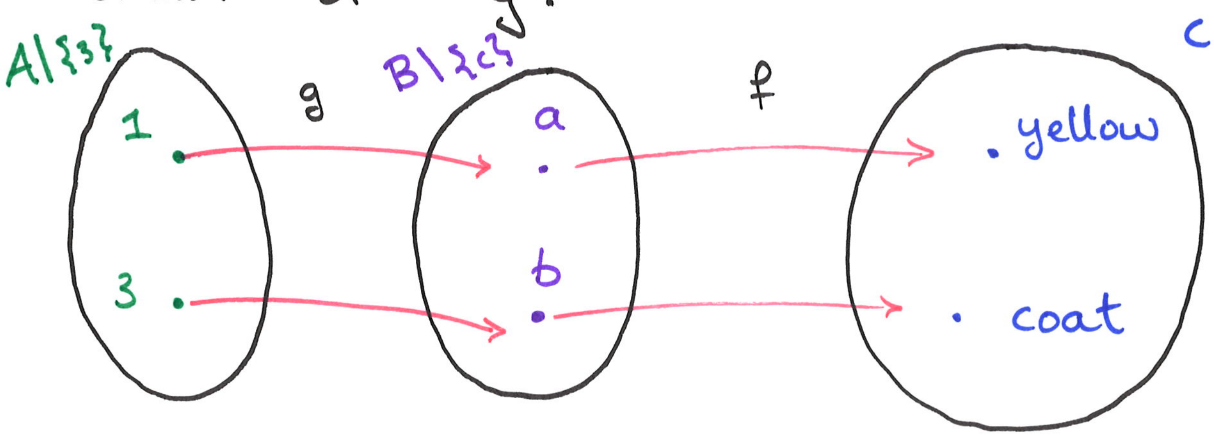
Answer: No. At least not at first.

What would $(f \circ g)(2)$ be?

$$2 \longrightarrow \boxed{g} \longrightarrow c \longrightarrow \boxed{f} \longrightarrow ???$$

c is not
in the domain
of f!

To make sense of $f \circ g$, we must shrink the domain of g:



$$f \circ g : A \setminus \{3\} \longrightarrow C$$

$$: \quad 1 \longmapsto \text{yellow}$$

$$: \quad 3 \longmapsto \text{coat}$$

$\underline{\underline{\text{NB}}}$ In general, the domain of $f \circ g$ is all x * such that $g(x)$ is in the domain of f.

(* : x in the domain of g)

(3) $g : \mathbb{R} \longrightarrow \mathbb{R}$

$\quad : x \longmapsto x+4 \quad (\text{i.e. } g(x) = x+4)$

$f : [0,\infty) \longrightarrow \mathbb{R}$

$\quad : x \longmapsto \sqrt{x} \quad (\text{i.e. } f(x) = \sqrt{x})$

Remark: We need to be careful when composing:

$$-5 \longrightarrow \boxed{g} \longrightarrow -1 \longrightarrow \boxed{f} \longrightarrow \sqrt{-1} \text{ ???}$$

Not in the domain of f!

Question: For what $x \in \mathbb{R}$ is $g(x)$ in the domain of f?

i.e. For what $x \in \mathbb{R}$ is $x+4$ in $[0,\infty)$?

i.e. For what $x \in \mathbb{R}$ is $0 \leq x+4 < \infty$?

Answer: $-4 \leq x < \infty$

i.e. for $\underline{x \in [-4,\infty)}$

This is the domain of $f \circ g$!

Conclusion:

$$f \circ g : [-4, \infty) \longrightarrow \mathbb{R}$$

$$: x \longmapsto f(g(x)) = \sqrt{g(x)^2} = \sqrt{x+4^2}$$